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Tachyon potentials, star products and universality

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Abstract: We develop an efficient recursive method to evaluate the tachyon potential using the relevant universal subalgebra of the open string star algebra. This method, using off-shell versions of Virasoro Ward identities, avoids explicit computation of conformal transformations of operators and does not require a choice of background. We illustrate the procedure with a pedagogic computation of the level six tachyon potential in an arbitrary gauge, and the evaluation of a few simple star products. We give a background independent construction of the so-called identity of the star algebra, and show how it fits into family of string fields generating a commutative subalgebra.

Keywords: Bosonic Strings, D-branes, Conformal and W Symmetry.

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1. Introduction

In the last few months there has been a resurgence of interest in string field theory. In particular, open string field theory has provided a direct approach to study the physics of string theory tachyons. This includes the case of tachyons living on the D-branes of bosonic string theory, and the case of tachyons living on non-BPS D-branes or on D-brane anti-D-brane pairs of superstring theories.

The old problem of the tachyonic instability in bosonic open string theory has been put into a novel perspective by Sen's conjecture that there is an extremum of the tachyon potential at which the total negative potential energy exactly cancels the tension of the D-brane [1]. Moreover, solitonic lump solutions of the tachyon effective potentials are identified with lower-dimensional branes [2, 1]. Similar conjectures exist for the tachyon living on a coincident D-brane anti-D-brane pair, or on non-BPS D-branes of type IIA or IIB superstring theories [3]–[8]. String field theory has provided precise quantitative tests of these conjectures [9]–[20]. Indeed, the level expansion scheme has turned open string field theory into a powerful computational tool. This scheme is based on the 'experimental' realization that by truncating the string field to its low-lying modes (keeping only the Fock states with $L_0 \leq l$), one obtains an approximation that gets more accurate as the level l is increased [21, 9].

For definiteness we focus here on the cubic open bosonic string field theory [22]. In order to compute an (off-shell) term coupling three space-time fields φ_i in the string field action, one must evaluate the correlator of the associated (typically non-primary) CFT operators $\mathcal{O}_i$ on a specific three punctured disk. Being cubic, the only additional terms in the action are quadratic and much easier to compute. In other string field theories there are higher order terms, and our discussion will apply with minor modifications.

There are two main algorithms that have been applied for such computations. In one method, one first computes the (complicated) finite conformal transformations needed to insert with appropriate local coordinates the (non-primary) vertex operators $\mathcal{O}_i$, and then evaluates the correlators using the OPE's of various two dimensional fields. An alternative algorithm makes use of an operator representation of the vertex as an object $\langle V_3|$ in the 3-string (dual) Fock space [23]–[26]. The desired correlator is given by the contraction $\langle V_3|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle$, where $|\mathcal{O}_i\rangle$ denotes the Fock space state associated to $\mathcal{O}_i$. This contraction requires purely algebraic manipulations and it can be naturally automated on a computer [11]. The explicit expression for $\langle V_3|$, however, is tied to the specific choice of CFT background. This approach, therefore, does not implement the background independence feature of tachyon condensation. The tachyon potential only involves string fields that correspond to CFT states built by acting on the vacuum with the universal oscillators $\{c_n, b_n\}$ and the (negatively moded) background independent matter Virasoro operators L_n^{matt} (with $c = 26$) [27]. We denote the subspace generated by these as $\mathcal{H}_{\text{univ}}$. A main objective of this paper is to provide a direct computational scheme using $\mathcal{H}_{\text{univ}}$.

Our procedure is the systematic off-shell implementation of the conventional Virasoro Ward Identities that allow the computation of correlators of Virasoro descendents in terms of those of Virasoro primaries. We view the vertex $\langle V_3|$ as an object where a negatively moded Virasoro operator in one state space can be converted into linear combinations of positively moded Virasoro operators in all state spaces, with readily calculable coefficients that capture the geometry of the interaction. Such relations allow recursive computation of all correlations involving the Virasoro operators. These relations are simply useful versions of the conservation laws of the operator formalism [28, 29]. They are obtained, for the Virasoro case, by studying contour integrals of the type $\int T(z)v(z)dz$ where $v(z)$ is a globally defined vector field on the punctured surface defining the interaction vertex. The identities arise by contour deformation and by referring the objects inside the integrals to the coordinates chosen at the punctures. Particular cases of the conservation laws of the operator formalism have been used since very early times in string theory. For string interactions based on contact type interactions (as in light cone theories and classical closed string field theory) such relations have gone under the name of 'overlap conditions'. By implementing analogous conservation laws for the ghost sector, we obtain a computational scheme totally within $\mathcal{H}_{\rm univ}$. This makes the background independence of the tachyon computations completely manifest. While the ideas behind our approach are certainly not new, their applications are.

Computations using conservation laws are elegant and simple to carry out. As we shall illustrate, the computations involved in [9] become very straightforward. As opposed to the contraction method based on the explicit Fock construction of $\langle V_3|$ the method we discuss is naturally recursive, and having done computatations to some level, only marginal additional work is required to go one level higher. In addition, compared with the set of all Lorentz scalars, which is used in the best calculation to date of the tachyon potential [11], the $\mathcal{H}_{\rm univ}$ basis becomes more and more economical as the level is increased. We are thus led to believe the present method would allow even higher level calculations. We also hope that the more transparent geometric understanding that is gained with conservation laws will help find an exact closed form expression for the string field tachyon-condensate representing the stable stationary point of the tachyon potential.

With this goal in mind we begin some exploration of the structure of star products using conservation laws. The recognition [27] of the important role played by the background-independent subspace $\mathcal{H}^{(1)}_{\rm univ}$ of ghost number one states, prompts some questions of a more formal nature. The space $\mathcal{H}_{\rm univ}$, built just as $\mathcal{H}^{(1)}_{\rm univ}$ but containing states of all ghost numbers, is readily identified as a natural subalgebra of the full star algebra. Conservation laws make this manifest. Since ghost number simply adds under the star product, $\mathcal{H}^{(0)}_{\rm univ}$ is a subalgebra of $\mathcal{H}_{\rm univ}$. In open string field theory gauge parameters have ghost number zero, therefore $\mathcal{H}^{(0)}_{\rm univ}$ is a (the?) universal subalgebra of the open string gauge algebra. Associated to once-punctured disks Σ one has ghost number zero states, usually referred to as 'surface states', as they arise from Riemann surfaces. Since such states $*$-multiply to give surface states, the set $\mathcal{H}(\Sigma)$ of such states is a subalgebra of $\mathcal{H}^{(0)}_{\rm univ}$.

It is natural to ask whether the so-called 'identity' element $\mathcal{I}$ of the string field $*$-algebra is an element of $\mathcal{H}_{\rm univ}$. It is. In fact $\mathcal{I}\in\mathcal{H}(\Sigma)$. The identity is the state associated to a unit disk with local coordinates that cover all of its interior. As such, it can be written as an exponential of *total* (matter + ghost) Virasoro generators acting on the vacuum. This description appears to be new. A denumerable basis for $\mathcal{H}(\Sigma)$ is provided by $\mathcal{H}^{(0)}(L) = \mathrm{Span}[\{L^{\rm tot}\}|0\rangle]$, the set of all total-Virasoro descendents of the $\mathrm{SL}(2,\mathbb{R})$ vacuum. We point out that the $\mathrm{SL}(2,\mathbb{R})$ vacuum and the identity $\mathcal{I}$ belong to a family of 'wedge-like' surface states $\mathcal{H}_{\text{wedge}}$ of the CFT, each of which is associated to a 1-punctured disk with local coordinates that cover a wedge of a certain angle within the unit circle. The identity corresponds to an angle of 360° and the vacuum to 180°. This family of wedge states is closed under $*$-multiplication and forms a commutative algebra. All in all we have the following inclusion of (universal) subalgebras of the star algebra:

$$\mathcal{H}_{\text{wedge}} \subset \mathcal{H}^{(0)}(L) \subset \mathcal{H}^{(0)}_{\text{univ}} \subset \mathcal{H}_{\text{univ}} \,. \tag{1.1}$$

This paper is organized as follows. In section 2 we review various descriptions of the three open string vertex, as well as the universal description of the tachyon string field. As a comment on [27], we explain that the tachyon string field is spanned by the action of matter and ghost Virasoro operators on the zero momentum tachyon. In section 3 we discuss in detail the derivation of Virasoro conservation laws. We extend this to ghost fields and to dimension one (non-primary) currents in section 4. As an illustration of our methods, in section 5 we compute the open string tachyon potential to level (2,6). We do this without gauge fixing. In section 6 we discuss the identity element and the subalgebra of wedge states. We also compute some star products, among them the product of two zero-momentum tachyons. We offer some concluding remarks in section 7.

2. Open string field theory and the tachyon

Section 2.1 is intended as a review of well-known results. In section 2.2 we then present a new characterization of the 'universal' subspace of states $\mathcal{H}_{\text{univ}}$ relevant for the tachyon condensation problem, as the states obtained acting with matter *and* ghost Virasoro generators on the tachyon $c_1|0\rangle$.

2.1 Open bosonic string field theory revisited

The dynamical variable of bosonic open string field theory (OSFT) is the string field $|\Phi\rangle$, which contains a component field for every state in the first-quantized string Fock space. The first-quantized Fock space of the open string is just the state space $\mathcal{H}$ of the combined matter *and* ghost Conformal Field Theories (CFT's). This state space $\mathcal{H}$ can be broken up into subspaces of definite ghost number. We will be using conventions where the $\mathrm{SL}(2,\mathbb{R})$ vacuum $|0\rangle$ carries ghost number zero, the b ghost carries ghost number -1 and the c ghost carries ghost number $+1$. With these conventions, a general off-shell string field in OSFT corresponds to a state in $\mathcal{H}$ with ghost number $+1$. A state in $\mathcal{H}$ can be represented as a local field acting on the vacuum

$$|\Phi\rangle = \Phi(0)|0\rangle \,, \tag{2.1}$$

where $\Phi(x)$ is defined on the boundary of the worldsheet. We shall mainly be using conventions where the CFT is defined on the upper-half complex plane, with the boundary of the worldsheet mapped to real axis.[1]

The classical open string field theory action is a function from $\mathcal{H}$ to the real numbers and is given by

$$S = -\frac{1}{g^2}\left(\frac{1}{2}\langle\Phi, Q_B\Phi\rangle + \frac{1}{3}\langle\Phi,\Phi,\Phi\rangle\right), \tag{2.2}$$

where g is the open string coupling constant, Q_B is the BRST charge, and the 2- and 3-point vertices $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot,\cdot\rangle$ are defined in terms of CFT correlators.

For the 2-point vertex

$$\langle\Phi,\Psi\rangle \equiv \langle I\circ\Phi(0)\ \Psi(0)\rangle, \tag{2.3}$$

where $\langle\ \rangle$ on the right hand side represents the CFT correlator and I denotes the $\mathrm{SL}(2,\mathbb{R})$ map $I(z) = -1/z$. The symbol $f\circ\Phi(0)$, where f is a complex map, means the conformal transform of $\Phi(0)$ by f. For example if Φ is a dimension d primary field, then $f\circ\Phi(0) = f'(0)^d\Phi(f(0))$. If Φ is non-primary the transformation rule will be more complicated and involve extra terms with higher derivatives of f. The cubic vertex is given by

$$\langle\Phi_1,\Phi_2,\Phi_3\rangle \equiv \langle f_1\circ\Phi_1(0) f_2\circ\Phi_2(0) f_3\circ\Phi_3(0)\rangle, \tag{2.4}$$

where f_i are some specific conformal maps which are described below (see (2.11)). We shall also write

$$\langle\Phi_1,\Phi_2,\Phi_3\rangle \equiv \langle V_3\,|\Phi_1\rangle\otimes|\Phi_2\rangle\otimes|\Phi_3\rangle, \tag{2.5}$$

where $\langle V_3| \in \mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{H}^*$, is a machine that given three CFT states produces a real number. Another familiar way of presenting the cubic vertex is in terms of a $*$-product defined as:

$$\langle\Phi,\Phi_1 * \Phi_2\rangle \equiv \langle\Phi,\Phi_1,\Phi_2\rangle, \qquad \forall\Phi\,. \tag{2.6}$$

We can specify a three string vertex with a picture showing how the worldsheets of the three strings join together. For the SFT in hand [22], the picture is given

[1] It is often useful to think of states $|\Phi\rangle$ in terms of their Schröedinger representation, that is, as functionals on the configuration space of strings. Consider the unit half-disk in the upper-half plane, $\{|z|\leq 1, \mathrm{Im}\,z\geq 0\}$, with the vertex operator $\Phi(0)$ inserted at the origin. Impose standard open string boundary conditions for the fields ϕ_i of the CFT on the real axis (ϕ_i is a collective label for all matter and ghost fields), and impose some specific boundary conditions ϕ_b on the outer boundary $|z| = 1$. The path integral over ϕ_i on the interior of this half-disk with the boundary conditions ϕ_b held fixed produces some functional $\Psi_\Phi[\phi_b]$. This functional assigns a complex number to each string configuration on the unit half-circle, and is therefore the Schröedinger representation of the state $\Phi(0)|0\rangle$.

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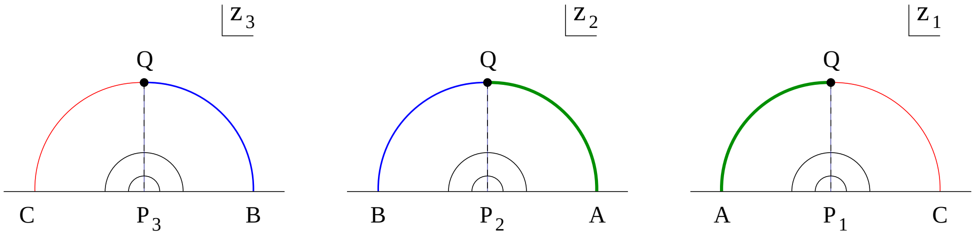

Figure 1: Representation of the cubic vertex as the gluing of 3 half-disks.

in figure 1. The worldsheets of the three strings are represented as unit half-disks $\{|z_i| \leq 1, \mathrm{Im}\, z \geq 0\}$, $i = 1, 2, 3$, in three copies of the complex plane. We glue the boundaries $|z_i| = 1$ of the three half-disks with the identifications:

$$\begin{aligned} z_1 z_2 &= -1\,, \qquad && \text{for } |z_1| = 1\,,\ \mathrm{Re}\ z_1 \leq 0 \\ z_2 z_3 &= -1\,, && \text{for } |z_2| = 1\,,\ \mathrm{Re}\ z_2 \leq 0 \\ z_3 z_1 &= -1\,, && \text{for } |z_3| = 1\,,\ \mathrm{Re}\ z_3 \leq 0\,. \end{aligned} \tag{2.7}$$

Note that the common interaction point Q, defined by $z_i = i$ (for $i = 1, 2, 3$) is the mid-point of each open string $|z_i| = 1$, $\mathrm{Im}\, z_i \geq 0$. The left half of the first string is glued with the right half of the second string, and the same is repeated cyclically. This construction defines a specific 'three-punctured disk', a genus zero Riemann surface with a boundary, three marked points (punctures) on this boundary, and a choice of local coordinates z_i around each puncture.

It will be useful to recognize that on this glued surface there is a globally defined Jenkins-Strebel quadratic differential [30]. This quadratic differential φ takes the form

$$\varphi = \phi(z_i) dz_i^2 = -\frac{1}{z_i^2}\, dz_i^2\,, \tag{2.8}$$

on each of the three coordinate patches. One readily verifies that this assignment is consistent with the identifications in (2.7). This quadratic differential has second order poles at the punctures $z_i = 0$. Its *horizontal trajectories*, the lines along which φ is real and positive, foliate the surface and represent the open strings. The quadratic differential has a first order zero at the interaction point Q ($z_i = i$). Indeed, three neighborhoods of angle π are being glued at Q and therefore a well-defined coordinate w at Q must be related to any given z_i as $w \sim (z_i - i)^{2/3}$. It then follows that near Q the quadratic differential takes the form $\varphi \sim dz_i^2 \sim w dw^2$, which shows the zero at $w = 0$. A quadratic differential defines a conformal metric $ds^2 = |\phi(z_i)||dz|^2$. With this metric the three half disks are presented as three semi-infinite strips of width π, as in figure 2. Gluing of these semi-infinite strips at the edges produces the concrete (metric) representation of the string vertex[2] (figure 3).

[2]This metric is actually a minimal area metric, this fact is important to understand why the Feynman rules of open string field theory generate a single cover of the moduli spaces of Riemann surfaces with boundaries [31, 32].

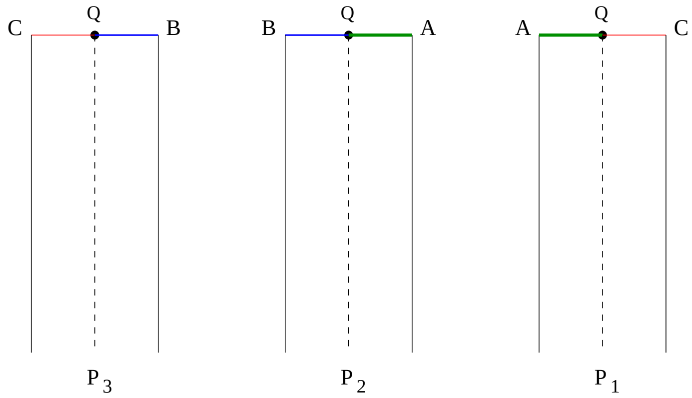

Figure 2: Representation of the cubic vertex as the gluing of 3 semi-infinite strips.

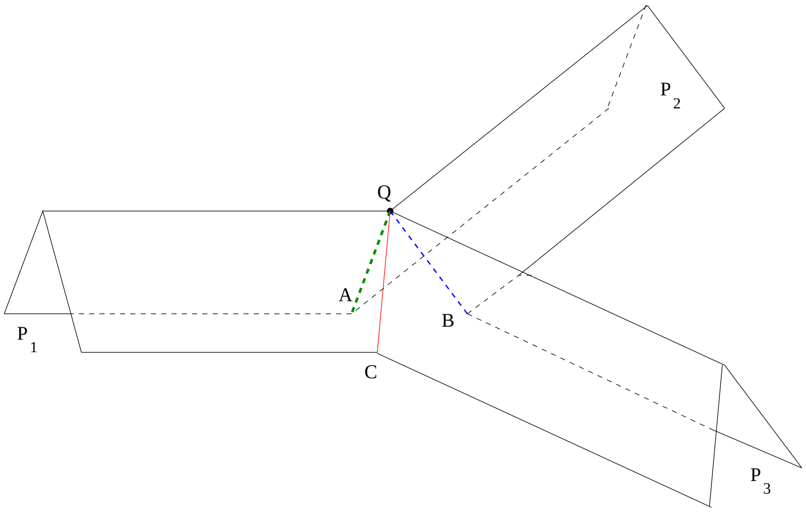

Figure 3: The result of gluing the 3 strips of figure 2.

Other conformal representations of the three string vertex are useful. For example, we can map the three half disks to the interior of the unit disk $|w| < 1$, as shown in figure 4. Each worldsheet is sent to a $120°$ wedge of this unit disk. To construct the explicit maps that send z_i to the w plane, one notices that the $\mathrm{SL}(2,\mathbb{C})$ transformation

$$h(z) = \frac{1+iz}{1-iz}\,, \tag{2.9}$$

maps the unit upper-half disk $\{|z| \leq 1, \mathrm{Im}\, z \geq 0\}$ to the 'right' half-disk $\{|w| \leq 1, \mathrm{Re}\, w \geq 0\}$, with $z = 0$ going to $w = h(0) = 1$. Thus the functions

$$\begin{aligned} F_1^{120°}(z_1) &= e^{\frac{2\pi i}{3}} \left(\frac{1+iz_1}{1-iz_1}\right)^{2/3}, \\ F_2^{120°}(z_2) &= \left(\frac{1+iz_2}{1-iz_2}\right)^{2/3}, \end{aligned}$$

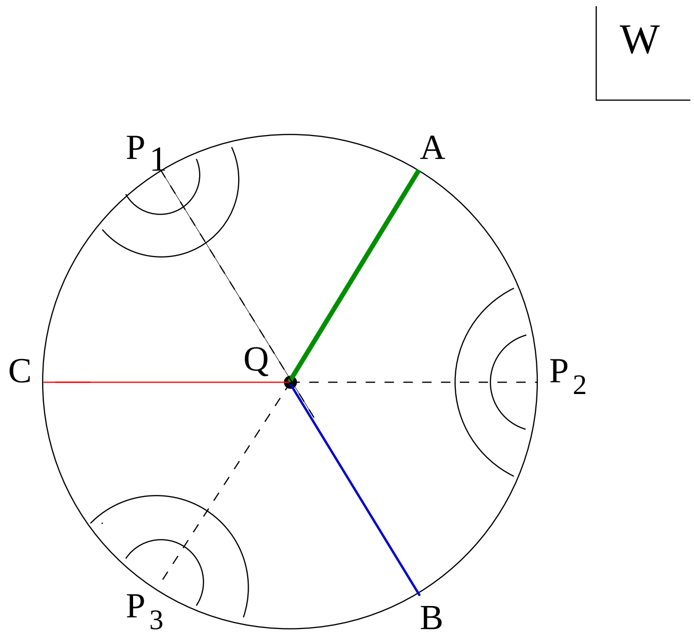

Figure 4: Representation of the cubic vertex as a 3-punctured unit disk.

$$F_3^{120°}(z_3) = e^{-\frac{2\pi i}{3}} \left(\frac{1+iz_3}{1-iz_3}\right)^{2/3} , \tag{2.10}$$

will send the three half-disks to three wedges in the w plane of figure 4, with punctures at $e^{\frac{2\pi i}{3}}$, 1, and $e^{-\frac{2\pi i}{3}}$ respectively. Identifying the functions f_i of (2.4) as $f_i \equiv F_i^{120°}$ we obtain a definition of the cubic vertex. In this representation cyclicity (i.e., $\langle \Phi_1, \Phi_2, \Phi_3 \rangle = \langle \Phi_2, \Phi_3, \Phi_1 \rangle$) is manifest by construction. By SL$(2,\mathbb{C})$ invariance, there are many other possible representations that give exactly the same off-shell amplitudes.

A useful choice is to map the interacting w disk symmetrically to the upper half plane. This is the convention that we shall mostly be using. We can therefore define the functions f_i by composing the earlier maps $F_i^{120°}$ (that send the half-disks to the w unit disk) with the map $h^{-1}(w) = -i\,\frac{w-1}{w+1}$ taking this unit disk to the upper-half-plane, with the three punctures on the real axis (figure 5),

$$\begin{aligned}
f_1(z_1) &\equiv h^{-1} \circ (F_1^{120°})(z_1) = S(f_3(z_1)) \\
&= \sqrt{3} + \frac{8}{3}\, z_1 + \frac{16}{9}\, \sqrt{3}\, {z_1}^2 + \frac{248}{81}\, {z_1}^3 + \frac{416}{243}\, \sqrt{3}\, {z_1}^4 + \frac{2168}{729}\, {z_1}^5 + O\left({z_1}^6\right). \\
f_2(z_2) &\equiv h^{-1} \circ (F_2^{120°})(z_2) = S(f_1(z_2)) = \tan\left(\frac{2}{3}\, \arctan(z_2)\right) \\
&= \frac{2}{3}\, z_2 - \frac{10}{81}\, {z_2}^3 + \frac{38}{729}\, {z_2}^5 + O\left({z_2}^7\right). \\
f_3(z_3) &\equiv h^{-1} \circ (F_3^{120°})\,(z_3) = S(f_2(z_3)) \qquad\qquad (2.11) \\
&= -\sqrt{3} + \frac{8}{3}\, z_3 - \frac{16}{9}\, \sqrt{3}\, {z_3}^2 + \frac{248}{81}\, {z_3}^3 - \frac{416}{243}\, \sqrt{3}\, {z_3}^4 + \frac{2168}{729}\, {z_3}^5 + O({z_3}^6).
\end{aligned}$$

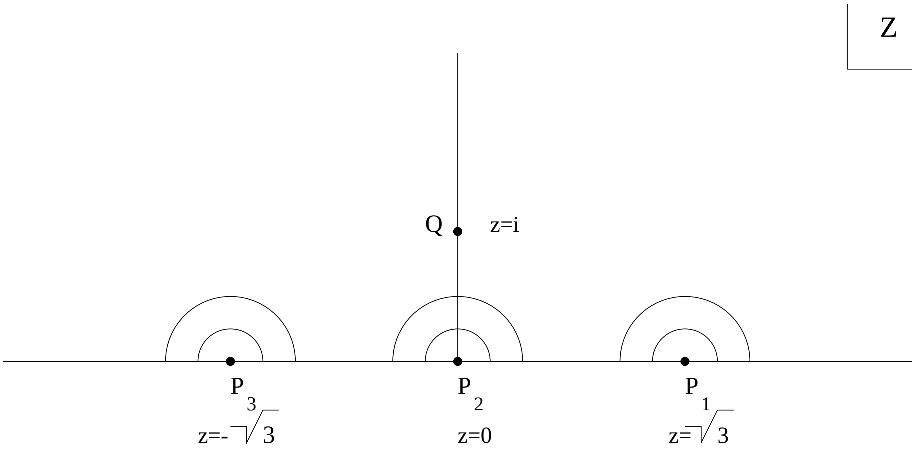

Figure 5: Representation of the cubic vertex as the upper-half plane with 3 punctures on the real axis.

The three punctures are at $f_1(0) = +\sqrt{3}$, $f_2(0) = 0$, $f_3(0) = -\sqrt{3}$, and the $\mathrm{SL}(2,\mathbb{R})$ map $S(z) = \frac{z-\sqrt{3}}{1+\sqrt{3}z}$ cycles them (thus $S \circ S \circ S(z) = z$). This completes the definition of the string field theory action.

2.2 The universal tachyon string field

The string field theory action (2.2) describes the dynamics of open strings in any conformal background. In particular, we can take the CFT to be the Boundary Conformal Field Theory (BCFT) of any bosonic Dp-brane, flat 26-dimensional Minkowski space being just the space filling D25-brane. Bosonic Dp-branes are unstable to decay and the open string tachyon is the signal of this instability. The zero momentum tachyon state is $c_1|0\rangle = c(0)|0\rangle$. This state belongs to a 'universal' subspace $\mathcal{H}_{\text{univ}}$ of $\mathcal{H}$ containing zero momentum scalars

$$\mathcal{H}_{\text{univ}} \equiv \text{Span}\{L^m_{-j_1} \dots L^m_{-j_p}\, b_{-k_1} \dots b_{-k_q}\, c_{-l_1} \dots c_{-l_r}|0\rangle\,,\; j_i \geq 2, k_i \geq 2, l_i \geq -1\}\,. \tag{2.12}$$

Note that the Virasoro generators belong to the matter CFT, and the restriction $j_i \geq 2$ arises because $L^m_{-1}|0\rangle = 0$. Similar restrictions apply to the ghost and antighost oscillators. This is a 'universal' subspace that is independent of the details of the BCFT (except for the central charge $c = 26$ of the matter stress-tensor). It is also clear that for a string field in $\mathcal{H}_{\text{univ}}$ the string field theory action (2.2) takes a background independent value, being determined by CFT correlators and operations that involve only the ghosts and the matter stress-tensor. This space $\mathcal{H}_{\text{univ}}$ is that introduced by Sen in [27], with the distinction that we are now including all ghost numbers.

It is remarkable that $\mathcal{H}_{\text{univ}}$ *defines a subalgebra of the star-algebra of open string fields.* While this follows as a simple extension of the logic in [27], we will give in section 6 a direct argument based on explicit construction of products in the level expansion. Note also that $Q_B : \mathcal{H}_{\rm univ} \to \mathcal{H}_{\rm univ}$, as the BRST charge Q_B is built from matter Virasoro and ghost oscillators. We can split $\mathcal{H}_{\rm univ}$ into a direct sum of spaces generated by states of a given ghost number:

$$\mathcal{H}_{\rm univ} = \oplus_{n\in Z}\mathcal{H}^{(n)}_{\rm univ}\,. \tag{2.13}$$

Since ghost number adds under the star multiplication we have that

$$\mathcal{H}^{(n)}_{\rm univ} * \mathcal{H}^{(m)}_{\rm univ} \subseteq \mathcal{H}^{(n+m)}_{\rm univ}\,, \tag{2.14}$$

where it follows that $\mathcal{H}^{(0)}_{\rm univ}$ is a closed subalgebra of $\mathcal{H}_{\rm univ}$. In section 6 we will discuss natural subalgebras of $\mathcal{H}^{(0)}_{\rm univ}$.

One of the main results of [27] is that the classical tachyon string field, a field that must clearly be at zero momentum and have ghost number one, can be assumed to lie on $\mathcal{H}^{(1)}_{\rm univ}$. All other fields can be set to zero consistently. We now claim that a basis for $\mathcal{H}^{(1)}_{\rm univ}$ is given by Fock states obtained acting on the tachyon $c_1|0\rangle$ with all matter *and* ghost Virasoro generators,

$$\mathcal{H}^{(1)}_{\rm univ} \equiv {\rm Span}\{L^m_{-j_1}\dots L^m_{-j_p}\, L^{gh}_{-l_1}\dots L^{gh}_{-l_p}\, c_1|0\rangle \;\; j_i\geq 2, l_i\geq 1\}\,. \tag{2.15}$$

The significance of this statement is that all computations of the tachyon potential reduce to computations of correlators of stress tensors and operations using the Virasoro algebra. The claim is equivalent to the statement that in the pure ghost CFT, all the states of ghost number one

$$\mathcal{G}^{(1)} \equiv {\rm Span}\{b_{k_1}\dots b_{k_q}\, c_{l_1}\dots c_{l_r}|0\rangle\;, r-q=1\}\,, \tag{2.16}$$

can be obtained as descendents of the tachyon $c_1|0\rangle$. Defining

$$V^{(1)} \equiv {\rm Span}\{L^{gh}_{-k_1}\dots L^{gh}_{-k_m} c_1|0\rangle, k_i\geq 1, k_1\leq k_2\dots\leq k_m\}\,, \tag{2.17}$$

to be the Verma module in the ghost CFT built on the primary $c_1|0\rangle$, we are claiming that $\mathcal{G}^{(1)} = V^{(1)}$. We can break up both spaces into subspaces of definite L_0 eigenvalue, which we denote by $V^{(1)}_n$ and $\mathcal{G}^{(1)}_n$ for $L_0 = n$. Clearly $V^{(1)}_n \subset \mathcal{G}^{(1)}_n$, so it suffices to show that $\dim V^{(1)}_n = \dim \mathcal{G}^{(1)}_n$ for each n. Let us first prove that the proposed basis states in (2.17) are linearly independent. If a linear combination $|\psi\rangle$ of descendents of $c_1|0\rangle$ vanished identically, $|\psi\rangle$ would be a null state of the CFT (a trivial one), and it should show up in the Kac determinant formula. The roots of the Kac determinant are given by

$$c = 1 - \frac{6}{m(m+1)}\,, \qquad h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}\,. \tag{2.18}$$

For $c=-26$, $m=-1/2\pm\sqrt{17/36}$. Since $c_1|0\rangle$ has $L_0=-1$, we are looking for integers r,s such that $h_{r,s}=-1$, and it is readily seen that there are no solutions.

Hence there are no null states in the Verma module of $c_1|0\rangle$, and the basis states in (2.17) are linearly independent. It only remains to show that the dimensions are the same.[3] We can find the dimensions of $V_n^{(1)}$ from the partition function

$$\sum x^n \dim V_n^{(1)} = \frac{1}{x}\prod_{k=1}^{\infty}\frac{1}{1-x^k}\,. \tag{2.19}$$

On the other hand we can determine the dimensions of the subspaces $\mathcal{G}_n^{(k)}$ of ghost number k and $L_0 = n$ by counting the ways we can act on the vacuum with the oscillators b_{-j} and c_{-m}, where $j \geq 2$ and $m \geq -1$. Indeed

$$\sum_{n,k} x^n y^k \dim \mathcal{G}_n^{(k)} = \prod_{j=2}^{\infty}\prod_{m=-1}^{\infty}\left(1+\frac{x^j}{y}\right)(1+y\,x^m) = \prod_{l=1}^{\infty}\frac{1}{1-x^l}\sum_{r=-\infty}^{+\infty} y^r x^{\frac{r^2-3r}{2}}\,, \tag{2.20}$$

where in the second equality we have used Jacobi triple product identity (see e.g. eq. (3.2.104) of [33]). We see that the term linear in y gives an expression for $\sum_n x^n \dim \mathcal{G}_n^{(1)}$ which precisely matches (2.19). This concludes the argument that all the states of ghost number one in the purely ghost CFT can be identified with the Verma module of $c_1|0\rangle$.[4]

3. Virasoro conservation laws

A straightforward procedure to evaluate the string field theory interaction vertex (2.4) is to compute the finite conformal transforms of the three vertex operators Φ_1, Φ_2, Φ_3, and then evaluate the CFT correlator using the relevant OPE's. Finite conformal transforms, however, are complicated for non-primary operators and this method quickly becomes very cumbersome at higher level. Conservation laws provide an elegant and efficient alternative. Let us illustrate this for the important case of generators of the Virasoro algebra, with central charge c. We shall derive identities obeyed by the string vertex $\langle V_3|$ of the general form

$$\langle V_3|L_{-k}^{(2)} = \langle V_3|\left(A^k\cdot c + \sum_{n\geq 0} a_n^k L_n^{(1)} + \sum_{n\geq 0} b_n^k L_n^{(2)} + \sum_{n\geq 0} c_n^k L_n^{(3)}\right), \tag{3.1}$$

where A^k, a_n^k, b_n^k and c_n^k are coefficients that will be determined below and depend on the geometry of the vertex. (By cyclicity, the same identity holds after letting

[3] The small argument below is familiar from bosonization of the ghost coordinates.

[4] At ghost number 0 there are states in the ghost CFT that are *not* Virasoro descendents of the vacuum $|0\rangle$ (the lowest dimensional primary at this ghost number). The simplest example is $|\chi\rangle = b_{-2}c_1|0\rangle = j^g(0)|0\rangle$, where j^g is the ghost number current. This state is not a primary since $j^g(z)$ is not a true tensor, indeed $L_1^{gh}|\chi\rangle = |0\rangle$. However $|\chi\rangle$ is not a descendent either, since $L_{-1}^{gh}|0\rangle = 0$. This phenomenon can occur in a non-unitary CFT.

(1) → (2), (2) → (3), (3) → (1)). The point of this identity is that the negatively moded Virasoro generator $L^{(2)}_{-k}$ acting on state space 2 is traded for a sum of positively moded generators acting on all the three state spaces, plus a central term. Since all the states in the background-independent subspace $\mathcal{H}^{(1)}_{\rm univ}$ are of the form (2.15), we see that by the conservation laws for matter and ghost Virasoro generators, and the commutation relations of the Virasoro algebra, we obtain a recursive procedure that allows one to express the coupling of any three states in $\mathcal{H}^{(1)}_{\rm univ}$ in terms of the coupling $\langle c_1, c_1, c_1\rangle$ of three tachyons.

3.1 Setting up Virasoro conservations

It is convenient to use the standard 'doubling trick' for open strings. We trade the holomorphic and antiholomorphic components of the stress tensor, defined in the upper-half z plane, for a single holomorphic field $T(z)$ defined in the whole complex plane. With this convention, the cubic vertex is regarded as a 3-punctured sphere. We examine a stress-tensor with general central term,

$$T(z')T(z) \sim \frac{c/2}{(z'-z)^4} + \frac{2\,T(z)}{(z'-z)^2} + \frac{\partial T(z)}{z'-z} + \cdots \tag{3.2}$$

Under holomorphic change of variables,

$$\tilde{T}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\, S(z,w)\,, \qquad S(z,w) = \frac{\frac{dz}{dw}\,\frac{d^3z}{dw^3} - \frac{3}{2}\left(\frac{d^2z}{dw^2}\right)^2}{\left(\frac{dz}{dw}\right)^2}\,. \tag{3.3}$$

The schwartzian derivative $S(z,w)$ vanishes if z and w are related by an $\mathrm{SL}(2,\mathbb{C})$ transformation. Under composition of conformal maps, $z \to \rho(z)$, $\rho \to w(\rho)$, one finds $S(w,z) = (d\rho/dz)^2\, S(w,\rho) + S(\rho,z)$.

Consider the representation (2.4), (2.11) of the cubic vertex in the full complex plane with punctures at $+\sqrt{3}$, 0 and $-\sqrt{3}$ (figure 5). We shall label the coordinate in the global plane as z, and the local coordinates around the punctures as z_i, $1 = 1,2,3$. Let $v(z)$ be a holomorphic vector field $v(z)$, thus transforming as $\tilde{v}(w) = (dzdw)^{-1}v(z)$. We require $v(z)$ to be holomorphic everywhere in the z plane, except at the punctures where it may have poles. Since in our convention the punctures are all located at finite points on the real axis, we need to impose regularity at infinity. Performing the change of variables $w = -1/z$, $\tilde{v}(w) = z^{-2}v(z)$. Hence for $v(z)$ to be regular at infinity, $\lim_{z\to\infty} z^{-2}v(z)$ must be constant (or zero).

The purpose of considering the vector $v(z)$ is that the product $v(z)T(z)dz$ transforms as a 1-form (except for a correction due to the central term),

$$T(z)v(z)\,dz = \Big(\tilde{T}(w) - \frac{c}{12}\,S(z,w)\Big)\tilde{v}(w)\,dw\,, \tag{3.4}$$

and can be naturally integrated along 1-cycles. Moreover this 1-form is *conserved*, thanks to the holomorphy of $T(z)$ and $v(z)$, and integration contours in the z plane can be continuously deformed as long as we do not cross a puncture. Consider a contour $\mathcal{C}$ which encircles the three punctures at $-\sqrt{3}$, 0 and $\sqrt{3}$ in the z plane of figure 5. For arbitrary vertex operators Φ_i, the correlator

$$\left\langle \oint_{\mathcal{C}} v(z)T(z)dz \; f_1 \circ \Phi_1(0) f_2 \circ \Phi_2(0) f_3 \circ \Phi_3(0) \right\rangle \tag{3.5}$$

vanishes identically, by shrinking the contour $\mathcal{C}$ to zero size around the point at infinity (which is a regular point). In this argument it is important that under the inversion $w = -1/z$, the schwartzian derivative vanishes and thus there is no contribution from the central term in (3.4). Since the correlator (3.5) is zero for arbitrary Φ_i, we can write

$$\langle V_3| \oint_{\mathcal{C}} v(z)T(z)dz = 0\,. \tag{3.6}$$

Deforming the contour $\mathcal{C}$ into the sum of three contours $\mathcal{C}_i$ around the three punctures, and referring the 1-form to the local coordinates, we obtain the basic relation

$$\langle V_3| \sum_{i=1}^{3} \oint_{\mathcal{C}_i} dz_i \; v^{(i)}(z_i) \left(T(z_i) - \frac{c}{12} S(f_i(z_i), z_i) \right) = 0\,. \tag{3.7}$$

The maps f_i, since they differ by $\mathrm{SL}(2,\mathbb{R})$ transformations, have the same schwartzian derivative. We find[5]

$$S(f_i, z_i) = -\frac{10}{9}\frac{1}{(1+z_i^2)^2} = -\frac{10}{9} + \frac{20}{9}z_i^2 - \frac{10}{3}z_i^4 + \frac{40}{9}z_i^6 + \cdots\,, \qquad i = 1,2,3\,. \tag{3.8}$$

Since this expression is regular at each puncture ($z_i = 0$), the central term can contribute to the conservation law (3.7) only for vector fields $v^{(i)}$ that have poles at the punctures.

We are looking for conservations laws of the form (3.1). Recalling that

$$L_{-k}^{(i)} = \frac{1}{2\pi i} \oint dz_i \; z_i^{-k+1} T^{(i)}(z_i)\,, \tag{3.9}$$

we need a vector field which behaves as $v^{(2)} \sim z_2^{-k+1} + O(z_2)$ around puncture 2, and has a zero in the other two punctures, $v^{(1)} \sim O(z_1)$, $v^{(3)} \sim O(z_3)$. A vector field of this type has (for $k > 1$) a pole at the second puncture, and is regular around the other two punctures. Contributions from the central term will then only appear in the second state space.

[5] For this and most explicit computations it is useful to use a symbolic manipulator such as Maple or Mathematica.

3.2 The first few conservation laws

Consider using the globally defined vector field

$$v_1(z) = -\frac{2}{9}\left(z^2-3\right). \tag{3.10}$$

As discussed before, this has zero at punctures 1 and 3 and is regular at infinity. Using the transformation law $v_1^{(i)}(z_i) = v_1(z(z_i))/(dz/dz_i)$ and the relations (2.11) we derive the Taylor expansion of the vector field referred to the each of the local coordinates

$$\begin{aligned}
v_1^{(1)}(z_1) &= -\frac{4}{3\sqrt{3}}z_1 + \frac{8}{27}{z_1}^2 - \frac{40}{81\sqrt{3}}\,{z_1}^3 + \frac{40}{729}{z_1}^4 + \frac{104}{729\sqrt{3}}\,{z_1}^5 + O\left({z_1}^6\right)\\
v_1^{(2)}(z_2) &= 1 + \frac{11}{27}{z_2}^2 - \frac{80}{729}{z_2}^4 + \frac{1136}{19683}{z_2}^6 + O\left({z_2}^8\right)\\
v_1^{(3)}(z_3) &= \frac{4}{3\sqrt{3}}z_3 + \frac{8}{27}{z_3}^2 + \frac{40}{81\sqrt{3}}\,{z_3}^3 + \frac{40}{729}{z_3}^4 - \frac{104}{729\sqrt{3}}\,{z_3}^5 + O\left({z_3}^6\right)
\end{aligned} \tag{3.11}$$

In this case the $v^{(i)}$ are regular around *each* puncture, so we get no contribution from the central term. Using (3.7) and noting that integration amounts to the replacement $v_n^{(i)} z_i^n \to v_n^{(i)} L_{n-1}^{(i)}$, we can immediately write the conservation law

$$\begin{aligned}
0 = \langle V_3| &\Big(-\frac{4}{3\sqrt{3}}\,L_0 + \frac{8}{27}L_1 - \frac{40}{81\sqrt{3}}L_2 + \frac{40}{729}L_3 + \frac{104}{729\sqrt{3}}L_4\cdots\Big)^{(1)} + \\
+ \langle V_3| &\Big(L_{-1} + \frac{11}{27}L_1 - \frac{80}{729}L_3 + \frac{1136}{19683}L_5 + \cdots\Big)^{(2)} + \\
+ \langle V_3| &\Big(\frac{4}{3\sqrt{3}}\,L_0 + \frac{8}{27}L_1 + \frac{40}{81\sqrt{3}}L_2 + \frac{40}{729}L_3 - \frac{104}{729\sqrt{3}}L_4\cdots\Big)^{(3)}.
\end{aligned} \tag{3.12}$$

Thanks to the cyclicity of the string vertex, analogous identities hold by cycling the punctures, (1) → (2), (2) → (3), (3) → (1). Using the vector field

$$v_2(z) = -\frac{4}{27}\,\frac{z^2-3}{z}\,, \tag{3.13}$$

we obtain

$$\begin{aligned}
0 = \langle V_3| &\Big(-\frac{8}{27}\,L_0 + \frac{80}{81\sqrt{3}}\,L_1 - \frac{112}{243}L_2 + \frac{304}{729\sqrt{3}}L_3 - \frac{400}{19683}L_4\cdots\Big)^{(1)} + \\
+ \langle V_3| &\Big(L_{-2} + \frac{5}{54}\,c + \frac{16}{27}\,L_0 - \frac{19}{243}L_2 + \frac{800}{19683}L_4 + \cdots\Big)^{(2)} + \\
+ \langle V_3| &\Big(-\frac{8}{27}\,L_0 - \frac{80}{81\sqrt{3}}\,L_1 - \frac{112}{243}L_2 - \frac{304}{729\sqrt{3}}L_3 - \frac{400}{19683}L_4\cdots\Big)^{(3)}.
\end{aligned} \tag{3.14}$$

Since $v_2(z)$ has a pole at puncture 2 we got a contribution from the central term.

In general we can get conservation laws for $L^{(2)}_{-k}$ with $v_k(z) \sim (z^2-3)z^{-k+1}$. For $k > 2$, using this vector in (3.7) one obtains an identity that besides $L^{(2)}_{-k}$ involves other negatively moded Virasoro generators $L^{(2)}_{-k+2}$, $L^{(2)}_{-k+4}$, ... It is straightforward to remove these terms by subtracting the conservation laws for smaller k. Indeed, using the vectors

$$\begin{aligned} v_3(z) &= -\frac{8}{81}\,\frac{z^2-3}{z^2} - \frac{7}{9}\,v_1(z)\,, \\ v_4(z) &= -\frac{16}{243}\,\frac{z^2-3}{z^3} - \frac{26}{27}\,v_2(z)\,, \end{aligned} \tag{3.15}$$

we obtain the conservation laws:

$$\begin{aligned} 0 = \langle V_3| &\Big(\frac{68}{81\sqrt{3}}L_0 + \frac{40}{243}L_1 - \frac{152}{243\sqrt{3}}L_2 + \frac{8792}{19683}L_3 - \frac{3320}{6561\sqrt{3}}L_4 \cdots\Big)^{(1)} + \\ &+ \langle V_3|\Big(L_{-3} - \frac{80}{243}L_1 + \frac{2099}{19683}L_3 - \frac{3568}{59049}L_5 + \cdots\Big)^{(2)} + \\ &+ \langle V_3|\Big(-\frac{68}{81\sqrt{3}}L_0 + \frac{40}{243}L_1 + \frac{152}{243\sqrt{3}}L_2 + \frac{8792}{19683}L_3 + \frac{3320}{6561\sqrt{3}}L_4 \cdots\Big)^{(3)}. \end{aligned} \tag{3.16}$$

$$\begin{aligned} 0 = \langle V_3| &\Big(\frac{176}{729}L_0 - \frac{416}{729\sqrt{3}}L_1 - \frac{800}{19683}L_2 + \frac{13280}{19683\sqrt{3}}L_3 - \frac{84448}{177147}L_4 \cdots\Big)^{(1)} + \\ &+ \langle V_3|\Big(L_{-4} - \frac{5}{27}c - \frac{352}{729}L_0 + \frac{1600}{19683}L_2 - \frac{8251}{177147}L_4 + \cdots\Big)^{(2)} + \\ &+ \langle V_3|\Big(\frac{176}{729}L_0 + \frac{416}{729\sqrt{3}}L_1 - \frac{800}{19683}L_2 - \frac{13280}{19683\sqrt{3}}L_3 - \frac{84448}{177147}L_4 \cdots\Big)^{(3)}. \end{aligned} \tag{3.17}$$

Higher identities can be derived analogously. The identities above suffice for a level (4,8) computation of the string action.

We conclude this section with a discussion of the so-called 'reparametrization invariances' of the cubic vertex [34]. It is well known that for $c = 0$ (*total* Virasoro generators) the combination

$$K_n = L_n - (-1)^n L_{-n}\,, \tag{3.18}$$

is conserved on the vertex, that is

$$\langle V_3|\left(K^{(1)}_n + K^{(2)}_n + K^{(3)}_n\right) = 0\,. \tag{3.19}$$

These relations are special cases of the Virasoro conservation laws and can be obtained by adding the 3 cyclic versions of the L_{-n} conservation. A direct and more elegant derivation is as follows. Consider the vector field defined by $v^{(i)}_n(z_i) = z_i^{n+1} - (-1)^n z_i^{-n+1}$ around each of the punctures. This vector field is globally defined since the expressions on each puncture are consistent with the gluing relations (2.7) of the string vertex. We have then

$$\langle V_3| \sum_{i=1}^{3} \oint T^{(i)}(z_i)\Big(z_i^{n+1} - (-1)^n z_i^{-n+1}\Big) = 0\,, \tag{3.20}$$

which immediately gives (3.19).[6]

4. Ghost and current conservation laws

The b ghost field is a true conformal tensor of dimension two. Thus its conservation laws are identical to those for the $c = 0$ Virasoro generators, with the formal replacement $L_k^{(i)} \to b_k^{(i)}$. In this section we derive conservation laws for the $c(z)$ field. In addition we derive conservation laws for currents.

4.1 Conservations for the c-ghost

The c ghost is a primary field of dimension minus one, $\tilde{c}(w) = (dz/dw)^{-1}c(z)$. To derive conservation laws, we consider a globally defined 'quadratic differential' on the sphere

$$\varphi = \phi(z)(dz)^2 = \phi'(z')(dz')^2\,, \tag{4.1}$$

holomorphic everywhere except for possible poles at the punctures. Regularity at infinity requires the $\lim_{z\to\infty} z^4\phi(z)$ to be finite. The product $c(z)\phi(z)dz$ is a 1-form, which is conserved thanks to holomorphy of $c(z)$ and $\phi(z)$. We can then use contour deformations and following exactly the same logic as for the 1-form $v(z)T(z)dz$ considered in the previous section, we obtain

$$\langle V_3| \sum_{i=1}^{3} \oint_{\mathcal{C}_i} dz_i\, c^{(i)}(z_i)\phi^{(i)}(z_i) = 0\,. \tag{4.2}$$

For example, with the quadratic differential

$$\phi_0(z) = -3\,z^{-2}(z^2-3)^{-1} \tag{4.3}$$

---

[6] Since the vector field $v_n^{(i)}$ was directly given in terms of the local coordinates z_i and shown to be globally defined the reader may wonder how the central term violations [23] of these identities would arise. Indeed, while the contour integrals can be canceled pairwise at the boundary of the local disks without extra contributions (the transition functions are projective), there is a subtlety at the interaction points (the points $z_i = \pm i$ on the local coordinates and $z = 0, \infty$ on the global disk). To deal with this properly one can cancel the contour integrals pairwise, but not all the way to the interaction points. This leaves three tiny contour integrals $\sum_i \int T(z_i)v(z_i)dz_i$ that add up to a contour surrounding each interaction point. To evaluate this one has to pass again to the coordinate z vanishing at the interaction point. A simple computation shows that the schwartzian $S(z_i, z)$ has a second order pole at $z = 0$. In addition, the vector v has a first order zero at $z = 0$. Thus a central charge contribution (in fact, of the right value) arises.

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one obtains:

$$0 = \langle V_3|\Big(-\frac{4}{3\sqrt{3}}\,c_1 + \frac{8}{27}\,c_2 + \frac{68}{81\sqrt{3}}\,c_3 - \frac{176}{729}\,c_4 + \cdots\Big)^{(1)} + \\ + \langle V_3|\Big(c_0 - \frac{16}{27}\,c_2 + \frac{352}{729}c_4 - \frac{8368}{19683}c_6 + \cdots\Big)^{(2)} + \\ + \langle V_3|\Big(\frac{4}{3\sqrt{3}}\,c_1 + \frac{8}{27}\,c_2 - \frac{68}{81\sqrt{3}}\,c_3 - \frac{176}{729}\,c_4 + \cdots\Big)^{(3)}. \tag{4.4}$$

Higher conservation laws are obtained with quadratic differentials having higher order poles at $z = 0$. With

$$\phi_1(z) = -2\,z^{-3}(z^2-3)^{-1}\,, \\ \phi_2(z) = -\frac{4}{3}\,z^{-4}(z^2-3)^{-1} + \frac{2}{9}\phi_0(z)\,, \tag{4.5}$$

we derive

$$0 = \langle V_3|\Big(-\frac{8}{27}\,c_1 + \frac{80}{81\sqrt{3}}\,c_2 - \frac{40}{243}\,c_3 - \frac{416}{729\sqrt{3}}\,c_4 + \cdots\Big)^{(1)} + \\ + \langle V_3|\Big(c_{-1} - \frac{11}{27}\,c_1 + \frac{80}{243}\,c_3 - \frac{5680}{19683}\,c_5 + \cdots\Big)^{(2)} + \\ + \langle V_3|\Big(-\frac{8}{27}\,c_1 - \frac{80}{81\sqrt{3}}\,c_2 - \frac{40}{243}\,c_3 + \frac{416}{729\sqrt{3}}c_4 + \cdots\Big)^{(3)}. \tag{4.6}$$

$$0 = \langle V_3|\Big(-\frac{40}{81\sqrt{3}}\,c_1 + \frac{112}{243}\,c_2 - \frac{152}{243\sqrt{3}}\,c_3 + \frac{800}{19683}\,c_4 + \cdots\Big)^{(1)} + \\ + \langle V_3|\,\Big(c_{-2} + \frac{19}{243}\,c_2 - \frac{1600}{19683}\,c_4 + \frac{4640}{59049}\,c_6 + \cdots\Big)^{(2)} + \\ + \langle V_3|\Big(\frac{40}{81\sqrt{3}}\,c_1 + \frac{112}{243}\,c_2 + \frac{152}{243\sqrt{3}}\,c_3 + \frac{800}{19683}\,c_4 + \cdots\Big)^{(3)}. \tag{4.7}$$

The linear combinations

$$C_n \equiv c_n + (-1)^n c_{-n}\;, \tag{4.8}$$

are analogous to the K_n's discussed at the end of the previous section, since they are conserved on the three string vertex,

$$\langle V_3|\,\big(C_n^{(1)} + C_n^{(2)} + C_n^{(3)}\big) = 0\,. \tag{4.9}$$

This identity is easily derived considering the quadratic differential defined by the *same* functional form

$$\phi_n^{(i)}(z_i) = z_i^{-2+n} + (-1)^n z_i^{-2-n} \qquad i = 1, 2, 3 \tag{4.10}$$

in the three local coordinate patches. These expressions are consistent with the gluing conditions (2.7). Thus this quadratic differential is globally defined and application of the basic relation (4.2) immediately gives (4.9). In particular for $n = 0$, $\phi_0^{(i)}$ is the canonical quadratic differential (2.8) on the string vertex, which gives the conservation law

$$\langle V_3| \left(c_0^{(1)} + c_0^{(2)} + c_0^{(3)}\right) = 0\,. \tag{4.11}$$

4.2 Current conservation

Let $j(z)$ be a holomorphic dimension one current. Moreover, assume that $j(z)$ is not primary and take the OPE of the stress tensor with the current to be:

$$T(z')\,j(z) = \frac{2q}{(z'-z)^3} + \frac{j(z)}{(z'-z)^2} + \frac{\partial\, j(z)}{(z'-z)} + \cdots\,, \tag{4.12}$$

where q is a dimensionless number. For example, $q = -3/2$ for the ghost number current of the (b,c) system. The finite transformation associated to $j(z)$ is

$$\frac{dz}{dw}\cdot j(z) = \tilde{\jmath}(w) - q\,\frac{d^2z}{dw^2}\left(\frac{dz}{dw}\right)^{-1}. \tag{4.13}$$

If $f(z)$ is an analytic *scalar* ($f(z) = \tilde{f}(w)$), than $j(z)f(z)dz$ transforms as a 1-form (except for a correction due to the anomaly q). We assume that $f(z)$ is holomorphic everywhere (including infinity) except for possible poles at the punctures and follow the usual strategy to derive conservation laws. Consider again a contour $\mathcal{C}$ which encircles counterclockwise the 3 punctures in the z plane of figure 5. This time $\langle V_3| \oint_{\mathcal{C}} dz\, j(z)f(z)$ is not identically zero, since to shrink the contour around the point at infinity we need to perform the transformation $z \to \tilde{z} = 1/z$, and the current is not covariant under this. One finds instead

$$\langle V_3| \left(\oint_{\mathcal{C}} dz\, j(z)f(z) + 2q\int_{\tilde{z}=0} d\tilde{z}\,\frac{\tilde{f}(\tilde{z})}{\tilde{z}}\right) = 0 \tag{4.14}$$

where $\tilde{z} = 1/z$. Here the second integral is evaluated along a contour that goes around the $\tilde{z}$ plane origin counterclockwise. This term will matter only when $\tilde{f}(\tilde{z})$ is regular or has poles at $\tilde{z} = 0$, in other words when $f(z)$ is regular or worse at infinity. We can now deform the contour $\mathcal{C}$ into the sum of 3 contours $\mathcal{C}_i$ around the 3 punctures and pass to the local coordinates. The conservation law reads

$$\langle V_3| \left(2q\int_{\tilde{z}=0} d\tilde{z}\,\frac{\tilde{f}(\tilde{z})}{\tilde{z}} + \sum_{i=1}^{3}\int\left[j(z_i) - q\,\frac{d^2z}{dz_i^2}\left(\frac{dz}{dz_i}\right)^{-1}\right]f^{(i)}(z_i)dz_i\right) = 0\,. \tag{4.15}$$

Note that there are two contributions to the anomaly.

The simplest conservation law arises for the scalar function $f_0(z) = 1$. The anomaly contribution arises from the first term in the above equation. We find

$$0 = \langle V_3|\left(j_0^{(1)} + j_0^{(2)} + j_0^{(3)} + 2q\right), \tag{4.16}$$

which reflects the familiar anomaly in charge conservation.

With the following functions

$$
\begin{aligned}
f_1(z) &= \frac{2}{3} \cdot \frac{1}{z}\,, \\
f_2(z) &= -\frac{4}{27}\,\frac{z^2-3}{z^2}\,, \\
f_3(z) &= -\frac{8}{81}\,\frac{z^2-3}{z^3} - \frac{11}{27}\,f_1(z)\,, \\
f_4(z) &= -\frac{16}{243}\,\frac{z^2-3}{z^4} - \frac{16}{27}\,f_2(z)\,,
\end{aligned}
\tag{4.17}
$$

we obtain the conservation laws:

$$
\begin{aligned}
0 = \langle V_3|\Big(\frac{2}{3\sqrt{3}}\,j_0 - \frac{16}{27}\,j_1 + \frac{32}{81\sqrt{3}}\,j_2 + \frac{16}{729}\,j_3 - \frac{64}{729\sqrt{3}}\,j_4 + \cdots\Big)^{(1)} + \\
+ \langle V_3|\Big(j_{-1} + \frac{5}{27}\,j_1 - \frac{32}{729}\,j_3 + \frac{416}{19683}\,j_5 + \cdots\Big)^{(2)} + \\
+ \langle V_3|\Big(-\frac{2}{3\sqrt{3}}\,j_0 - \frac{16}{27}\,j_1 - \frac{32}{81\sqrt{3}}\,j_2 + \frac{16}{729}\,j_3 + \frac{64}{729\sqrt{3}}\,j_4 + \cdots\Big)^{(3)}.
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
0 = \langle V_3|\Big(-\frac{64}{81\sqrt{3}}\,j_1 + \frac{128}{243}\,j_2 - \frac{320}{729\sqrt{3}}\,j_3 - \frac{256}{19683}\,j_4 + \cdots\Big)^{(1)} + \\
+ \langle V_3|\Big(j_{-2} + \frac{22}{27}q + \frac{2}{9}\,j_0 - \frac{13}{243}\,j_2 + \frac{512}{19683}\,j_4 + \cdots\Big)^{(2)} + \\
+ \langle V_3|\Big(\frac{64}{81\sqrt{3}}\,j_1 + \frac{128}{243}\,j_2 + \frac{320}{729\sqrt{3}}\,j_3 - \frac{256}{19683}\,j_4 + \cdots\Big)^{(3)}.
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
0 = \langle V_3|\Big(-\frac{22}{81\sqrt{3}}\,j_0 + \frac{16}{243}\,j_1 + \frac{160}{243\sqrt{3}}\,j_2 - \frac{10288}{19683}\,j_3 + \frac{3136}{6561\sqrt{3}}\,j_4 + \cdots\Big)^{(1)} + \\
+ \langle V_3|\Big(j_{-3} - \frac{32}{243}\,j_1 + \frac{893}{19683}\,j_3 - \frac{1504}{59049}\,j_5 + \cdots\Big)^{(2)} + \\
+ \langle V_3|\Big(\frac{22}{81\sqrt{3}}\,j_0 + \frac{16}{243}\,j_1 - \frac{160}{243\sqrt{3}}\,j_2 - \frac{10288}{19683}\,j_3 - \frac{3136}{6561\sqrt{3}}\,j_4 + \cdots\Big)^{(3)}.
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
0 = \langle V_3|\Big(\frac{256}{729\sqrt{3}}\,j_1 - \frac{512}{19683}\,j_2 - \frac{12544}{19683\sqrt{3}}\,j_3 + \frac{91136}{177147}\,j_4 + \cdots\Big)^{(1)} + \\
+ \langle V_3|\Big(j_{-4} - \frac{562}{729}q - \frac{38}{243}\,j_0 + \frac{1024}{19683}\,j_2 - \frac{5125}{177147}\,j_4 + \cdots\Big)^{(2)} + \\
+ \langle V_3|\Big(-\frac{256}{729\sqrt{3}}\,j_1 - \frac{512}{19683}\,j_2 + \frac{12544}{19683\sqrt{3}}\,j_3 + \frac{91136}{177147}\,j_4 + \cdots\Big)^{(3)}.
\end{aligned}
\tag{4.21}
$$

Both for j_{-2} and j_{-4} the anomaly receives contributions from the two terms in (4.15).

5. Sample computation: level six tachyon potential

For the purposes of illustration we will compute here the open string field action relevant to tachyon condensation to level $(2, 6)$. In order to provide not only an illustration but also new information, we will compute the action without imposing a gauge choice.

5.1 Notation and the basic three point function

We begin with some preliminaries. Given three vertex operators $\widehat{A}, \widehat{B}, \widehat{C}$ we denote the corresponding states by $A|0\rangle, B|0\rangle$, $C|0\rangle$, where A, B, C denote sets of oscillators. As explained before (see section 2.1),

$$\begin{aligned}\langle \widehat{A}, \widehat{B}, \widehat{C}\rangle &= \langle V_3|\, A^{(1)}|0\rangle_{(1)} \otimes B^{(2)}|0\rangle_{(2)} \otimes\, C^{(3)}|0\rangle_{(3)}\,,\\ &= \langle V_3|\, A^{(1)}B^{(2)}C^{(3)}\,|0\rangle_{(3)} \otimes |0\rangle_{(2)} \otimes\, |0\rangle_{(1)}\,,\end{aligned} \tag{5.1}$$

where in the second step we moved the vacua all the way to the right.[7] For Grassmann odd operators (the operators relevant to the classical action) the following cyclicity and twist relations hold

$$\langle \widehat{A}, \widehat{B}, \widehat{C}\rangle = \langle \widehat{B}, \widehat{C}, \widehat{A}\rangle\,, \qquad \langle \widehat{A}, \widehat{B}, \widehat{C}\rangle = \Omega_A\Omega_B\Omega_C\,\langle \widehat{C}, \widehat{B}, \widehat{A}\rangle\,, \tag{5.2}$$

where Ω is the twist eigenvalue and equals $(-)^{N+1}$, where N is the total number operator measured with respect to the $\mathrm{SL}(2,\mathbb{R})$ vacuum. Also with a little abuse of notation we will denote $\langle \widehat{A}, \widehat{B}, \widehat{C}\rangle$ simply by $\langle A, B, C\rangle$, where we simply write the Fock space oscillators instead of the operators.

The basic correlator that we will need is

$$\langle c_1, c_1, c_1\rangle = \langle V_3|\, c_1^{(1)}\, c_1^{(2)}\, c_1^{(3)}\, |0\rangle_{(3)} \otimes |0\rangle_{(2)} \otimes |0\rangle_{(1)} \tag{5.3}$$

For this correlator we use the conformal field theory definition and thus write it as:

$$\begin{aligned}\langle c_1, c_1, c_1\rangle &= \langle f_1\circ c(0), f_2\circ c(0), f_3\circ c(0)\rangle\\ &= \Big\langle\, \frac{c(\sqrt{3})}{8/3}\,, \frac{c(0)}{2/3}, \frac{c(-\sqrt{3})}{8/3}\Big\rangle\\ &= \frac{3^3}{2^7}\langle c(\sqrt{3})c(0)c(-\sqrt{3})\rangle = \frac{3^4\sqrt{3}}{2^6}\,.\end{aligned} \tag{5.4}$$

In obtaining this we used

$$f_i\circ c(0) = \frac{c(f_i(0))}{f_i'(0)}\,, \qquad \langle c(z_1)c(z_2)c(z_3)\rangle = (z_1-z_2)(z_1-z_3)(z_2-z_3)\,. \tag{5.5}$$

and equations (2.11) to read the values of the $f_i'(0)$. The answer is written as

$$\langle c_1, c_1, c_1\rangle = K^3\,, \qquad K \equiv \frac{3\sqrt{3}}{4}\,. \tag{5.6}$$

The conservation laws allow the computation of all necessary three point functions in terms of this single one. No other three point function need to be evaluated directly for the problem of finding the tachyon potential.

[7] In the operator formulation of bosonic open string field theory it is natural to use a Grassmann even string field. In this case, the in-vacuum $|0\rangle$ is Grassmann odd. The rearrangement of the vacua in the second equation is convenient to avoid explicit signs.

5.2 Level (2,6) tachyon potential in arbitrary gauge

The relevant level two string field[8] is then:

$$|T\rangle = tc_1|0\rangle + uc_{-1}|0\rangle + vL^m_{-2}\,c_1\,|0\rangle + w\,c_0b_{-2}c_1|0\rangle\,. \tag{5.7}$$

Note that w is the only field that would not have appeared in the Siegel gauge. We will now compute the string field theory action for such a string field. Even though the action is not gauge fixed, gauge invariance is actually broken by the level truncation. The tachyon potential $\mathcal{V}(T)$ we will compute is given by

$$\frac{\mathcal{V}(T)}{2\pi^2 M} \equiv \frac{1}{2\pi^2} f(T) = V(T) = \frac{1}{2}\langle T, Q_B T\rangle + \frac{1}{3}\,\langle T, T * T\rangle\,, \tag{5.8}$$

where M denotes the brane mass.

The computation of the kinetic terms requires the BRST operator. To the required level, and at zero momentum this is given by

$$\begin{aligned} Q_B = {} & c_0\,L_0^{\rm tot} - 2\,b_0c_{-1}c_1 - 4\,b_0c_{-2}c_2 - \\ & -3\,b_{-1}c_{-1}c_2 - 3c_{-2}c_1b_1 + L^m_{-2}\,c_2 + c_{-2}L^m_2\,. \end{aligned} \tag{5.9}$$

In addition, the rules of BPZ conjugation for mutually (anti)commuting oscillators is simply to let $\phi_n \to (-)^{n+h}\phi_{-n}$ where h is the conformal dimension of the world-sheet field ϕ. In considering a product of such oscillators the order is not reversed.[9] Thus for example: $bpz(c_1|0\rangle) = \langle 0|c_{-1}$, and $bpz(c_0b_{-2}c_1|0\rangle) = -\langle 0|c_0b_2c_{-1}$. With these relations and using $\langle 0|c_{-1}c_0c_1|0\rangle = 1$, the computation of the kinetic terms is straightforward. We find:

$$\frac{1}{2\pi^2} f(T)_{\rm kin} = -\frac{1}{2}\,t^2 - \frac{1}{2}\,u^2 + \frac{13}{2}\,v^2 + 2\,w^2 + 3\,u\,w - 13\,v\,w\,. \tag{5.10}$$

Let us now consider the computation of the various cubic couplings. It will be instructive to consider the possible use of different basis states to describe the modes of the string field. Given the relations

$$\begin{aligned} c_{-1}|0\rangle &= \frac{1}{2}L^2_{-1}\,c_1|0\rangle\,, \\ c_0b_{-2}c_1|0\rangle &= \Big(\frac{1}{2}L^{gh}_{-2} - \frac{3}{4}L^2_{-1}\Big)c_1\,|0\rangle\,, \end{aligned} \tag{5.11}$$

[8] We have changed the normalization of the v field by ommitting a $\sqrt{13}$ that was included in [9].

[9] In more general cases BPZ conjugation is dealt with by using a reflector state representing a two punctured disk (or sphere) with coordinates z and $-1/z$ around 0 and ∞ respectively. The conservation laws for such state are of the type $\langle R_{12}|(\phi^{(1)}_n - (-)^{n+h}\phi^{(2)}_{-n}) = 0$. When having a Fock space state $|\mathcal{O}\rangle_{(1)}$ made by the product of (possibly non-commuting) oscillators living in state space (1), the BPZ conjugate is simply defined to be $\langle R_{12}|\mathcal{O}\rangle_{(1)}$. In evaluating this state one uses the conservation laws until all oscillators of $\mathcal{O}$ live in state space (2) at which point the vacuum in state space (1) deletes one of the punctures leaving the out vacuum in state space (2): $\langle R_{12}|0\rangle_{(1)} = {}_{(2)}\langle 0|$. The final BPZ state is a bra. Note that for non-commuting oscillators the final ordering is the reversed one.

we can express all the above states in the basis consisting of ghost and matter Virasoro descendents of the tachyon. The computation can then be carried out using conservation laws for matter ($c = 26$) and ghost ($c = -26$) Virasoro operators. This procedure generalizes to computations to arbitrary level, since as shown in section 2.2 ghost and matter Virasoro descendents of the tachyon form a basis of $\mathcal{H}^{(1)}_{\rm univ}$.

Alternatively, we can also use the ghost U(1) current $j(z) =: c(z)b(z) := \sum \frac{j_n}{z^{n+1}}$. Here $q = -3/2$, j_0 is ghost number, and $[j_n, j_m] = n\,\delta_{n+m,0}$. The tachyon state is a U(1) primary, namely $j_n\, c_1|0\rangle = 0$ for $n \geq 1$. A simple computation gives

$$\begin{aligned} c_{-1}|0\rangle &= \frac{1}{2}\Big(j_{-2} + j_{-1}j_{-1}\Big)c_1|0\rangle\,, \\ c_0 b_{-2}c_1|0\rangle &= \frac{1}{2}\Big(j_{-2} - j_{-1}j_{-1}\Big)c_1|0\rangle\,. \end{aligned} \tag{5.12}$$

Let us illustrate the various possibilities with the computation of the ttu term in the action. The factor of (1/3) in front of the cubic term cancels with the symmetry factor of three, which is the number of ways the t, t, u fields can be assigned to three punctures. Given the cyclicity property the position of u is irrelevant. We therefore have to compute

$$\langle c_1, c_{-1}, c_1\rangle = \langle V_3|\, c_1^{(1)}\, c_{-1}^{(2)}\, c_1^{(3)}\, |0\rangle_{(3)} \otimes |0\rangle_{(2)} \otimes |0\rangle_{(1)}\,, \tag{5.13}$$

where the subscripts denote the labels distinguishing the three state spaces. More precisely we want to relate this term to the tachyon one given in (5.3). To this end we can simply use the conservation law for c_{-1} as given in (4.6). Note that the only term that contributes is described by the replacement $c_{-1}^{(2)} \to \frac{11}{27}c_1^{(2)}$. There are no extra signs since the sign necessary to bring c_{-1} next to the vertex bra is compensated by the sign needed to bring the reflected oscillator c_1 back to the middle position. Therefore

$$\langle c_1, c_{-1}, c_1\rangle = \frac{11}{27}\langle c_1, c_1, c_1\rangle = \frac{11}{27}K^3\,. \tag{5.14}$$

Therefore the corresponding term in the potential is

$$V(T) = \cdots + \frac{11}{27}K^3 utt\,. \tag{5.15}$$

We can repeat this calculation describing $c_{-1}|0\rangle$ as a $U(1)$ ghost current descendent. To this end we note that $q = -3/2$ and compute

$$\langle c_1, j_{-2}c_1, c_1\rangle = \Big(-\frac{22}{27}\,q - \frac{2}{9}\Big)\langle c_1, c_1, c_1\rangle = K^3\,, \tag{5.16}$$

upon use of (4.19) and noticing that reflected operators into the first and third state spaces do not contribute. Similarly

$$\langle c_1, j_{-1}j_{-1}c_1, c_1\rangle = -\frac{5}{27}\langle c_1, j_1 j_{-1}c_1, c_1\rangle = -\frac{5}{27}\,K^3\,. \tag{5.17}$$

The two last results, combined using the first equation in (5.12), agree with (5.14). Using the ghost recursion relations one readily finds the additional terms:

$$\langle c_1, c_{-1}, c_{-1}\rangle = \frac{19}{243}\,K^3\,, \qquad \langle c_{-1}, c_{-1}, c_{-1}\rangle = \frac{1}{81}\,K^3\,. \tag{5.18}$$

Incorporating such terms into the potential we now have

$$V(T) = \cdots + \Big(\frac{11}{27}\,utt + \frac{19}{243}tuu + \frac{1}{243}uuu\Big)K^3\,. \tag{5.19}$$

Let us now consider some correlators involving only matter operators. For example, to compute the ttv interaction we write

$$\langle c_1, L^m_{-2}c_1, c_1\rangle = \langle \mathbf{1}, L^m_{-2}\mathbf{1}, \mathbf{1}\rangle_m \cdot \langle c_1, c_1, c_1\rangle \tag{5.20}$$

since the ghost and matter correlators factorize. In the first correlator the normalization is $\langle \mathbf{1},\mathbf{1},\mathbf{1}\rangle_m = 1$. The matter correlator is readily computed using the conservation of L_{-2}

$$\langle \mathbf{1}, L_{-2}\mathbf{1}, \mathbf{1}\rangle = -\frac{5}{54}\,c\,, \tag{5.21}$$

where in the present application $c = 26$. Therefore we have

$$\langle c_1, L^m_{-2}c_1, c_1\rangle = -\frac{65}{27}K^3\,. \tag{5.22}$$

Using the Virasoro recursion relations one readily finds the following expressions, valid for arbitrary c:

$$\begin{aligned}
\langle \mathbf{1}, L_{-2}\mathbf{1}, \mathbf{1}\rangle &= -\frac{5}{54}\,c \equiv f_{200}(c)\,,\\
\langle L_{-2}\mathbf{1}, L_{-2}\mathbf{1}, \mathbf{1}\rangle &= \frac{128}{729}\,c + \frac{25}{2916}\,c^2 \equiv f_{220}(c)\,,\\
\langle L_{-2}\mathbf{1}, L_{-2}\mathbf{1}, L_{-2}\mathbf{1}\rangle &= \frac{4096}{19683}\,c - \frac{320}{6561}\,c^2 - \frac{125}{157464}\,c^3 \equiv f_{222}(c)\,.
\end{aligned} \tag{5.23}$$

These equations are easily used to produce all couplings of v to tachyons:

$$\begin{aligned}
V(T) &= \cdots + \Big(f_{200}(26)\,vtt + f_{220}(26)\,vvt + \frac{1}{3}f_{222}(26)\,vvv\Big)K^3\\
&= \cdots + \Big(-\frac{65}{27}\,vtt + \frac{7553}{729}\,vvt - \frac{272363}{19683}\,vvv\Big)K^3\,.
\end{aligned} \tag{5.24}$$

As a last illustration consider couplings to the field w, outside the Siegel gauge. Using the b_{-2} conservation (read from that of L_{-2} with $c = 0$) we have

$$\begin{aligned}
\langle c_1, c_0b_{-2}c_1, c_1\rangle &= -\langle V_3|\, c_1^{(1)}\, b_{-2}^{(2)}c_0^{(2)}c_1^{(2)}\, c_1^{(3)}\, |0\rangle_{(3)} \otimes |0\rangle_{(2)} \otimes |0\rangle_{(1)}\\
&= +\frac{16}{27}\,\langle V_3|\, c_1^{(1)}\, b_0^{(2)}c_0^{(2)}c_1^{(2)}\, c_1^{(3)}\, |0\rangle_{(3)} \otimes |0\rangle_{(2)} \otimes |0\rangle_{(1)}\\
&= +\frac{16}{27}\,\langle V_3|\, c_1^{(1)}\, c_1^{(2)}\, c_1^{(3)}\, |0\rangle_{(3)} \otimes |0\rangle_{(2)} \otimes |0\rangle_{(1)} = \frac{16}{27}K^3\,.
\end{aligned} \tag{5.25}$$

This gives the ttw coefficient. The tuw coefficient requires

$$\langle c_{-1}, c_0 b_{-2} c_1, c_1 \rangle = \frac{16}{81} K^3 , \tag{5.26}$$

which again, is most effectively done by using first the b_{-2} conservation. Other coefficients are computed similarly. It is perhaps worth noting that the coefficient of w^3 vanishes because of the conservation $\langle V_3 | (c_0^{(1)} + c_0^{(2)} + c_0^{(3)}) = 0$.

Collecting together our results we find that the interacting part of the potential is given by

$$\begin{aligned} \frac{1}{2\pi^2} f^{(4)}_{\text{inter}}(T) = \Big(& \frac{1}{3} t^3 + \frac{11}{27} t^2 u - \frac{65}{27} t^2 v + \frac{16}{27} t^2 w + \\ & + \frac{19}{243} t u^2 + \frac{7553}{729} t v^2 + \frac{64}{243} t w^2 - \\ & - \frac{1430}{729} t u v - \frac{2080}{729} t v w - \frac{32}{81} t u w + \\ & + \frac{1}{243} u^3 - \frac{272363}{19683} v^3 + 0 \cdot w^3 - \\ & - \frac{1235}{6561} u^2 v + \frac{83083}{19683} u v^2 - \frac{11248}{19683} u^2 w + \frac{120848}{19683} v^2 w - \\ & - \frac{3008}{19683} u w^2 - \frac{4160}{6561} v w^2 + \frac{2080}{2187} u v w \Big) K^3 . \end{aligned} \tag{5.27}$$

The string field theory is gauge invariant, but a level truncated expression will not be so. While the gauge transformations are nonlinear and mix all fields, to the linearized level they take the form $\delta|\Phi\rangle = Q_B |\Lambda\rangle$. With gauge parameter $\epsilon b_{-2} c_1 |0\rangle$ we find

$$\delta w = \epsilon , \qquad \delta u = 3\epsilon , \qquad \delta v = \epsilon ,$$

and therefore $(u - 3w)$ and $(v - w)$ are gauge invariant at the linearized level. While the Siegel gauge $w = 0$ is possible, other choices may be allowed. In the level expansion one should not expect all gauge choices to give the same results at each level. Allowed gauge choices should give the same results for physical questions in the infinite level limit. While there is no proof that the Siegel gauge is a good gauge non-pertubatively, the results obtained in tachyon computations certainly suggest this is the case.

One can try to find a critical point for the above potential without using any gauge. Had the potential been gauge invariant one would expect undetermined parameters, but since gauge invariance is broken by the truncation, one finds a definite critical point.[10] The critical point is found at $t = 0.573$, $u = -0.1074$, $v = -0.04225$, $w = -0.1807$, and gives 88.0% of the required vacuum energy. In contrast, in the Siegel gauge, we obtain about 96% of the vacuum energy (with $t = 0.544, u = 0.190, v = 0.0560$).

[10] Ashoke Sen has suggested that this uncontrolled lifting of flat directions is likely to make numerical work based on gauge invariant actions less reliable. One could certainly find large fluctuations in the expectation values associated with quasi-flat directions.

6. The identity, star products and universality

In this section we use the technology developed before to get insight into the nature of the identity string field. We also learn how to compute explicitly in the level expansion a few star products, including that of two zero-momentum tachyons. We isolate a family of string fields associated to once-punctured disks. Such surface states, called *wedge states* because the local coordinate half-disk defines a wedge of the unit disk, form a subalgebra of the star algebra. Finally, we show using conservation laws that $\mathcal{H}_{\text{univ}}$ defines a subalgebra of the star algebra.

6.1 The so-called identity string field

A formal object which has often been considered in discussions of open string field theory is the identity element of the $*$ algebra, a string field $|\mathcal{I}\rangle$ which is formally expected to obey (with some caveats that will be discussed below)

$$|\mathcal{I}\rangle * |\Phi\rangle = |\Phi\rangle * |\mathcal{I}\rangle = |\Phi\rangle \tag{6.1}$$

for any string field $|\Phi\rangle$. A Fock space expression for $|\mathcal{I}\rangle$ in terms of flat-space free-field oscillators has been given in [23]. It is natural to ask whether $|\mathcal{I}\rangle$ can be given a manifestly background-independent representation.

The answer is immediate when we recall the geometric interpretation of the identity. In the Schröedinger representation, the functional $\Psi_{\mathcal{I}}[\phi(\sigma)]$ associated to the identity acts on an open-string configuration $\phi(\sigma)$ (ϕ is a collective name for the matter and ghost fields) as a delta function overlap of the left and right halves of the string. In other terms, $\Psi_{\mathcal{I}}[\phi(\sigma)]$ vanishes unless the left ($0 \leq \sigma \leq \pi/2$) and right ($\pi/2 \leq \sigma \leq \pi$) halves of the string exactly coincide. In CFT language, such functional is represented by a state $\langle\mathcal{I}| \in \mathcal{H}^*$ satisfying $\langle\mathcal{I}|\,\Phi\rangle \equiv \langle\Phi\rangle$ for all $|\Phi\rangle \in \mathcal{H}$, where the one point function is computed on a specific 1-punctured disk, one where the local coordinates are such that the left and right halves of the worldsheet boundary are glued together. $\langle\mathcal{I}|$ is a surface state, since just as $\langle V_3|$ it encodes the correlators on a particular Riemann surface.

Representing as usual the worldsheet as the upper half-disk $\{|z| \leq 1, \operatorname{Im} z \geq 0\}$, the function

$$w = F^{360^\circ}(z) = \left(\frac{1+iz}{1-iz}\right)^2 \tag{6.2}$$

sends this upper-half unit disk to the full unit disk in the w plane, mapping the two halves of the string ($\{|z| = 1, \operatorname{Re} z > 0\}$ and $\{|z| = 1, \operatorname{Re} z < 0\}$) to the same interval $\{\operatorname{Im} w = 0, -1 \leq \operatorname{Re} w \leq 0\}$. It is convenient to map back the disk to the upper half-plane. Our final choice of local coordinate is then

$$\tilde{z} = f^{360^\circ}(z) = h^{-1}(F^{360^\circ}(z)) = \frac{2z}{1-z^2}\,, \tag{6.3}$$

where h was defined in (2.9). The puncture is at $\tilde{z} = 0$, and the image of the unit upper half disk is the full upper half $\tilde{z}$-plane.

An explicit representation for the identity is now easy to write. If we can find the operator $U_{f^{360°}}$ that implements the conformal transformation $f^{360°}(z)$ in the CFT state space, we have

$$\langle \mathcal{I}| = \langle 0|\, U_{f^{360°}}\,. \tag{6.4}$$

Such operator must be written as the exponential of a linear combination of the total (matter + ghost) Virasoro generators [26]

$$U_f = e^{v_0 L_0} e^{\sum_{n\geq 1} v_n L_n}\,. \tag{6.5}$$

This makes manifest the background independence of $\langle \mathcal{I}|$. Since $f(0) = 0$ only positively moded Virasoro generators enter U_f, and we have chosen to separate out the global scaling component $e^{v_0 L_0}$. The coefficients v_n can be determined recursively from the Taylor expansion of f, by requiring [26, 35]

$$\begin{aligned} e^{v_0} &= f'(0) \\ \exp\left(\sum_{n\geq 1} v_n z^{n+1}\partial_z\right) z &= [f'(0)]^{-1} f(z) = z + a_2 z^2 + a_3 z^3 + \cdots\,. \end{aligned} \tag{6.6}$$

For example, for the first coefficients one finds

$$v_1 = a_2\,, \qquad v_2 = -a_2^2 + a_3\,, \qquad v_3 = \frac{3}{2}\,a_2^3 - \frac{5}{2}\,a_2 a_3 + a_4\,. \tag{6.7}$$

Taking $f = f^{360°}$ we obtain

$$U_{f^{360°}} = 2^{L_0}\exp\Big(L_2 - \frac{1}{2}L_4 + \frac{1}{2}L_6 - \frac{7}{12}L_8 + \frac{2}{3}L_{10} + \cdots\Big)\,. \tag{6.8}$$

By SL$(2,\mathbb{R})$ invariance of the vacuum, $\langle 0|L_0 = 0$, so that $\langle \mathcal{I}|$ does not in fact depend on the overall scaling factor v_0.[11]

To obtain the ket $|\mathcal{I}\rangle$ simply recall that BPZ conjugation sends L_n to $(-1)^n L_{-n}$,

$$|\mathcal{I}\rangle = \exp\Big(L_{-2} - \frac{1}{2}L_{-4} + \frac{1}{2}L_{-6} - \frac{7}{12}L_{-8} + \frac{2}{3}L_{-10} + \cdots\Big)|0\rangle\,. \tag{6.9}$$

This expression is well-defined in the level truncation scheme. Only a finite number of terms in the sum $\sum_n v_n L_{-n}$ are needed to write the expression of $|\mathcal{I}\rangle$ truncated at some given level.[12]

[11] Incidentally, SL$(2,\mathbb{R})$ invariance also guarantees that the surface state is independent of the particular SL$(2,\mathbb{R})$ frame used to write the map f. If $\tilde{f} = R\circ f$, where R is an SL$(2,\mathbb{R})$ transformation, then $\langle 0|U_{\tilde{f}} = \langle 0|U_R U_f = \langle 0|U_f$. The composition law $U_{g\circ f} = U_g U_f$ holds in a CFT with vanishing central charge, as is the case here.

[12] This property is not immediately obvious for the Fock space expression of [23].

The formalism of conservation laws allows to deduce several properties of the identity. Let us begin with Virasoro conservation laws. Let $\tilde{z}$ be as in (6.3) the global coordinate on the 1-punctured disk (upper half plane) associated to $\langle\mathcal{I}|$, and z the local coordinate around the puncture. For any vector field $\tilde{v}(\tilde{z})$, holomorphic everywhere (including infinity), except for a possible pole at the puncture $\tilde{z}=0$ we have

$$\langle\mathcal{I}|\oint_{\mathcal{C}}\tilde{T}(\tilde{z})\tilde{v}(\tilde{z})d\tilde{z}=0\,, \tag{6.10}$$

where $\mathcal{C}$ is a contour circling the origin. As usual, this statement is obtained by shrinking the contour to the point at infinity. By passing to the local coordinate z we deduce

$$\langle\mathcal{I}|\oint_{\mathcal{C}}dz\,v(z)\Big(T(z)-\frac{c}{12}S(f^{360^\circ}(z),z)\Big)=0\,. \tag{6.11}$$

We have $S(f^{360^\circ}(z),z)=6(z^2+1)^{-2}=6(1-2z^2+3z^4-4z^6+\cdots)$. Taking $\tilde{v}(\tilde{z})=\tilde{z}$, $\tilde{v}(\tilde{z})=2$, and $\tilde{v}(\tilde{z})=4/\tilde{z}$, respectively, we find

$$\begin{aligned}\langle\mathcal{I}|\,(L_0-2L_2+2L_4-2L_6+\cdots)&=0\,,\\ \langle\mathcal{I}|\,(L_{-1}-3L_1+4L_3-4L_5+\cdots)&=0\,,\\ \langle\mathcal{I}|\,(L_{-2}-(c/2)-4L_0+7L_2-8L_4+\cdots)&=0\,.\end{aligned} \tag{6.12}$$

One can use (6.9) to check these conservation laws in the level expansion. In addition, for $c=0$ Virasoro generators we also have

$$\langle\mathcal{I}|K_n=0\,. \tag{6.13}$$

This is proved as follows. We consider the vector field $v(z)=z^{n+1}-(-)^n z^{-n+1}$ as before and confirm that: (i) it satisfies $v(-1/z)z^2=v(z)$ as required by the identification $z\to-1/z$, (ii) the vector field has no poles anywhere else in the $\tilde{z}$ plane. Checking (ii) requires verifying this is the case both for $\tilde{z}=i$ (in fact, $v(\tilde{z}=i)$ is zero)) and for $\tilde{z}=\infty$ (in fact for $\tilde{z}\to\infty$, $v(\tilde{z})\sim\tilde{z}^2$, which is regular at $\tilde{z}\to\infty$).

Let us now consider the action of ghost oscillators on the identity. While the above argumentation suggests conservation laws based on the C_n operators (see (4.8), (4.9)) the quadratic differential (4.10) would actually have poles at $\tilde{z}=i$ invalidating the conservation. We therefore start from basics, the conservations take the form

$$\langle\mathcal{I}|\oint_{\mathcal{C}}dz\,\phi(z)c(z)=0\,, \tag{6.14}$$

where $\phi(z)$ is a quadratic differential holomorphic everywhere except at the puncture, and z is again the local coordinate defined in (6.3). Consider quadratic differentials of the form $\tilde{\phi}(\tilde{z})=1/\tilde{z}^n$, manifestly regular at $\tilde{z}=i$. Since the quadratic differential must not have a pole at $\tilde{z} = \infty$ we must require $n \geq 4$. Let us consider the simplest one:

$$\tilde{\phi}(\tilde{z}) = \frac{1}{\tilde{z}^4} \;\rightarrow\; \phi(z) = \frac{(1+z^2)^2}{2z^4}\,. \tag{6.15}$$

Note that this quadratic differential is well-defined in the glued surface, as $\varphi = \phi(z)dz^2$ is invariant under the gluing that identifies z with $-1/z$ on the halves of $|z| = 1, \mathrm{Im}\, z > 0$. Back in (6.14) we get

$$\langle \mathcal{I}|\Big(c_0 + \frac{1}{2}(c_2 + c_{-2})\Big) = 0\,. \tag{6.16}$$

It follows from this conservation law that $c_0|\mathcal{I}\rangle \neq 0$. This, in fact, shows that $|\mathcal{I}\rangle$ *is not an identity of the star algebra on all states.* Indeed, consider the value of $c_0(\mathcal{I} * A)$, which must equal $c_0 A$ if $\mathcal{I}$ is an identity for A. Using (4.11), however, we actually find that $c_0(\mathcal{I} * A) = c_0\mathcal{I} * A + \mathcal{I} * c_0 A$. If $\mathcal{I}$ is also an identity for $c_0 A$, it follows that $c_0\mathcal{I} * A = 0$ for all A. While this might actually be true for Fock space states A,[13] for $A = \mathcal{I}$ it implies that $c_0\mathcal{I}$ vanishes, which is patently false. Such failure of formal properties is certainly familiar in open string field theory [36].

Finally, we consider a holomorphic current $\tilde{j}(\tilde{z})$ and a holomorphic scalar $\tilde{f}(\tilde{z})$ and find

$$2q\,\langle \mathcal{I}| \oint_{\tilde{w}=0} d\tilde{w}\, \frac{\widehat{f}(\tilde{w})}{\tilde{w}} + \langle \mathcal{I}| \oint_{C} dz\, f(z) \left(j(z) - q\, \frac{d^2 f^{360^\circ}}{dz^2} \left(\frac{df^{360^\circ}}{dz}\right)^{-1}\right) = 0\,. \tag{6.17}$$

Here z is the local coordinate, $\tilde{z} = f^{360^\circ}(z) = 2z/(1-z^2)$ and $\tilde{w} = 1/\tilde{z}$. The functions $\widehat{f}$ and f are conformal tranforms of $\tilde{f}$ defined by the scalar tranformation laws $\widehat{f}(\tilde{w}) = \tilde{f}(\tilde{z})$ and $f(z) = \tilde{f}(\tilde{z})$. The simplest conservation arises for a constant $\tilde{f}$, giving $\langle \mathcal{I}|(2q + j_0) = 0$. For the BRST current $j_B(z)$, we have $j_B^0 = Q_B$, and $q = 0$. This gives

$$\langle \mathcal{I}|\, Q_B = 0\,. \tag{6.18}$$

This property is also manifest from the representation (6.9), since $[Q_B, L_n] = 0$ and $Q_B|0\rangle = 0$.

6.2 A subalgebra of wedge states

The observation that the identity is the surface state associated with the map F^{360° naturally leads us to consider a generalization to 'wedge-like' states of arbitrary angle. As we shall explain, this family of states, arising from once punctured disks, has the interesting property of being a subalgebra of the $*$-algebra.

We begin by considering the map

$$w = F^{\frac{360^\circ}{n}}(z) = \left(\frac{1+iz}{1-iz}\right)^{2/n}, \tag{6.19}$$

which sends the unit upper half disk in the z plane to a wedge of angle $360^\circ/n$ in

[13] We thank M. Schnabl for raising this possibility.

the w plane. Such maps are used in the higher point interactions of open superstring field theory [37]. As usual, we map back to the upper half plane:

$$\tilde{z} = f^{\frac{360^\circ}{n}}(z) = h^{-1}\left(F^{\frac{360^\circ}{n}}(z)\right) = \tan\left(\frac{2}{n}\arctan(z)\right). \tag{6.20}$$

Let us then define the family of surface states

$$\left\langle\frac{360^\circ}{n}\right| \equiv \langle 0|U_{f^{\frac{360^\circ}{n}}}\,. \tag{6.21}$$

With the same method used to calculate the identity we can write an explicit expression for these wedge states:

$$\begin{aligned}\left|\frac{360^\circ}{n}\right\rangle = \exp\Bigg(&-\frac{n^2-4}{3n^2}L_{-2} + \frac{n^4-16}{30n^4}L_{-4} - \\ &- \frac{(n^2-4)(176+128n^2+11n^4)}{1890n^6}L_{-6} + \cdots\Bigg)|0\rangle\end{aligned} \tag{6.22}$$

For $n = 1$ we recover the identity: $|360^\circ\rangle = |\mathcal{I}\rangle$ (see (6.9)). For $n = 2$ we get the vacuum: $|180^\circ\rangle = |0\rangle$. For $n \to \infty$ we find a smooth limit

$$\left|\frac{360^\circ}{\infty}\right\rangle = \exp\Big(-\frac{1}{3}L_{-2}+\frac{1}{30}L_{-4}-\frac{11}{1890}L_{-6}+\frac{1}{1260}L_{-8}+\frac{34}{467775}L_{-10}+\cdots\Big)|0\rangle\,. \tag{6.23}$$

The existence of the $n \to \infty$ limit can be understood from the expression for the conformal map,

$$f^{\frac{360^\circ}{n}}(z) \overset{n\to\infty}{\longrightarrow} \frac{2}{n}\arctan(z)\,. \tag{6.24}$$

The map has a well-defined limit up to a vanishing scaling factor, which is immaterial in the definition of the surface state.

The wedge surface states form a subalgebra of $\mathcal{H}^{(0)}_{\rm univ}$, in particular we claim that[14]

$$\left|\frac{360^\circ}{r_1}\right\rangle * \left|\frac{360^\circ}{r_2}\right\rangle = \left|\frac{360^\circ}{r_1+r_2-1}\right\rangle. \tag{6.25}$$

This is readily understood by sewing the appropriate surfaces. To sew wedges, however, we must first use conformal transformations of power type to eliminate corners. Let the input states above be defined on disks ξ_i with local coordinates η_i: $\xi_i(\eta_i) = F^{\frac{360^\circ}{r_i}}(\eta_i)$ $(i = 1, 2)$. We must then introduce new disks $w_i = (\xi_i)^{r_i/2}$. It then follows that $w_i(\eta_i) = F^{180^\circ}(\eta_i)$, and while the total neighborhood angle at $w_i = 0$ is πr_i, the image of the local η_i-half disk is a 180° wedge in w_i. These are now ready for sewing. On the three string vertex with disk w we introduce a new disk $W = w^{3/2}$ where each of the 120° wedges, the images of the local coordinates z_i, grows to a 180° wedge. The gluing necessary to take the product is simply $z_i\eta_i = -1 (i = 1, 2)$. This is simply implemented by amputating the two 180° wedges on the vertex W, amputating the two 180° wedges on w_i and gluing the left-overs of the w_i disks to

[14] Here r and s will denote real numbers larger than or equal to one.

the leftover of the W disk. The glued surface has a total neighborhood angle of $\pi(r_1-1)+\pi(r_2-1)+\pi=\pi(r_1+r_2-1)$, with the local coordinate giving the last contribution of π. Mapping back to a 360° disk, we see that this is simply the wedge state $|\frac{360°}{r_1+r_2-1}\rangle$.

More generally, we could consider the family of arbitrary surface states of the CFT, i.e., states of the form $\langle f|\equiv\langle 0|U_f$ for a generic conformal map f. The gluing theorem [26] guarantees that this family of string fields forms a closed subalgebra. The product $|f\rangle*|g\rangle$ is equal to another surface state $|h\rangle$, where the conformal map h is implicitly determined from f and g by the gluing procedure. It would be interesting to find a more explicit characterization of h.

6.3 Star products and $\mathcal{H}_{\rm univ}$

The wedge state $|\frac{360°}{3}\rangle = |120°\rangle$ has an interesting interpretation. Consider the $*$-product of two vacuum states. We immediately have

$$|0\rangle*|0\rangle=|180°\rangle*|180°\rangle=|120°\rangle\,, \tag{6.26}$$

where we made use of (6.25). This answer is easily understood if we recall that the $\mathrm{SL}(2,\mathbb{R})$ vacuum deletes punctures. It then follows from the disk presentation of the vertex (figure 4) that the resulting surface is a once punctured disk with a 120° wedge. Using the $n=3$ case of (6.22) we get:

$$|0\rangle*|0\rangle=\exp\left(-\frac{5}{27}L_{-2}+\frac{13}{486}L_{-4}-\frac{317}{39366}L_{-6}+\frac{715}{236196}L_{-8}+\cdots\right)|0\rangle\,. \tag{6.27}$$

At this time we can compute a couple of additional star products. Consider the evaluation of $|0\rangle*c_1|0\rangle$. The idea is to use a ghost conservation law to remove the c_1 from the state space where it appears and take it into the state space corresponding to the output. More precisely, in the expression

$$|0\rangle*c_1|0\rangle=\langle V_{123'}|\Big(|0\rangle_{(1)}\,\otimes\,|R_{3'3}\rangle\otimes c_1^{(2)}|0\rangle_{(2)}\Big) \tag{6.28}$$

we need a conservation law where the c_1 can be traded for oscillators that annihilate the vacuum in the first state space. Such conservation arises from the quadratic differential $\varphi(z)=\frac{9\sqrt{3}}{2}\,z^{-1}(z+\sqrt{3})^{-3}$:

$$\begin{aligned}0=\langle V_3|\Big(\frac{4}{3\sqrt{3}}\,c_2+\cdots\Big)^{(1)}+\langle V_3|\Big(c_1+\cdots\Big)^{(2)}-&\\-\langle V_3|\Big(\frac{27}{16}\,c_{-1}+\frac{3\sqrt{3}}{8}\,c_0-\frac{11}{16}\,c_1-\frac{2}{3\sqrt{3}}\,c_2+\frac{5}{9}\,c_3+\frac{44}{81\sqrt{3}}\,c_4+\cdots\Big)^{(3)}.\end{aligned} \tag{6.29}$$

Recalling that acting on the reflector $|R_{33'}\rangle$, we have $c_n^{(3)}\to c_{-n}^{(3')}(-)^{n+1}$ we obtain:

$$|0\rangle*c_1|0\rangle=\left(\frac{27}{16}c_1-\frac{3\sqrt{3}}{8}c_0-\frac{11}{16}c_{-1}+\frac{2}{3\sqrt{3}}c_{-2}+\frac{5}{9}c_{-3}-\frac{44}{81\sqrt{3}}c_{-4}\cdots\right)(|0\rangle*|0\rangle)\,. \tag{6.30}$$

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Note that this product manifestly lies on $\mathcal{H}_{\rm univ}$. Using the quadratic differential $\varphi(z) = 9\sqrt{3}\,(z-\sqrt{3})^{-1}(z+\sqrt{3})^{-3}$ we find

$$
\begin{aligned}
0 = \langle V_3|\Big(c_1 + \cdots\Big)^{(1)} + \langle V_3|\Big(-\frac{4}{3\sqrt{3}}c_2 + \cdots\Big)^{(2)} - \\
- \langle V_3|\left(\frac{27}{16}\,c_{-1} - \frac{3\sqrt{3}}{8}\,c_0 - \frac{11}{16}\,c_1 + \frac{2}{3\sqrt{3}}\,c_2 + \frac{5}{9}\,c_3 - \frac{44}{81\sqrt{3}}\,c_4 + \cdots\right)^{(3)} . \quad (6.31)
\end{aligned}
$$

This conservation law enables us to write

$$
c_1|0\rangle * c_1|0\rangle = \left(\frac{27}{16}c_1 + \frac{3\sqrt{3}}{8}c_0 - \frac{11}{16}c_{-1} - \frac{2}{3\sqrt{3}}c_{-2} + \frac{5}{9}c_{-3} + \frac{44}{81\sqrt{3}}\,c_{-4} + \cdot\cdot\right)(|0\rangle * c_1|0\rangle)\,. \quad (6.32)
$$

Combining the results above we can write

$$
\begin{aligned}
c_1|0\rangle * c_1|0\rangle = \;& \frac{81\sqrt{3}}{64}\left(c_0 - \frac{16}{27}\,c_{-2} + \frac{352}{729}\,c_{-4} + \cdots\right) \times \\
& \times \left(c_1 - \frac{11}{27}\,c_{-1} + \frac{80}{243}\,c_{-3} + \cdots\right)(|0\rangle * |0\rangle)\,. \qquad (6.33)
\end{aligned}
$$

This is a formula for the product of two zero momentum tachyons. One can recognize the two factors acting on $|0\rangle * |0\rangle$ as the factors that arise in the c_{-1} and c_0 conservations at the special punctures (see (4.4) and (4.6)).

The geometric interpretation of $|0\rangle * |0\rangle$ can also be used to find an alternative expression for $c_1|0\rangle * c_1|0\rangle$. This product is a $120°$ wedge, with puncture at $w = 1$, and two c's inserted at the other two punctures. This gives

$$
bpz(c_1|0\rangle * c_1|0\rangle) = \langle 0|c(e^{\frac{2\pi i}{3}})c(e^{-\frac{2\pi i}{3}})U_{F^{120°}}\left(\frac{dF_1^{120°}(0)}{dz}\frac{dF_3^{120°}(0)}{dz}\right)^{-1}, \qquad (6.34)
$$

or equivalently, using the upper-half plane representation of the vertex

$$
bpz(c_1|0\rangle * c_1|0\rangle) = \langle 0|c(\sqrt{3})c(-\sqrt{3})U_{f^{120°}}\left(\frac{df_1^{120°}(0)}{dz}\frac{df_3^{120°}(0)}{dz}\right)^{-1}. \qquad (6.35)
$$

The derivative factors arise from the conformal transformations of $c_1|0\rangle$, which is a primary field of dimension -1. We find

$$
\begin{aligned}
c_1|0\rangle * c_1|0\rangle &= \frac{9}{64}\cdot \exp\left(v_n^{120°}L_{-n}\right)\left(\frac{2}{3}\right)^{L_0} 3^2 c\left(\frac{1}{\sqrt{3}}\right)c\left(-\frac{1}{\sqrt{3}}\right)|0\rangle \\
&= \frac{27}{32}\cdot \exp\left(-\frac{5}{27}L_{-2} + \frac{13}{486}L_{-4} - \frac{317}{39366}L_{-6} + \frac{715}{236196}L_{-8} + \cdots\right) \times \\
&\quad \times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{2}{3\sqrt{3}}\right)^{2i+2j-1} c_{-2j}c_{-2i+1}|0\rangle\,. \qquad (6.36)
\end{aligned}
$$

Unlike the case of wedge states, the scaling component $(2/3)^{L_0}$ of $U_{f^{120°}}$ cannot be ignored.

By thinking of the evaluation of star products using the conservation laws one readily sees that $\mathcal{H}_{\rm univ}$ is a subalgebra, $\mathcal{H}_{\rm univ} * \mathcal{H}_{\rm univ} \subseteq \mathcal{H}_{\rm univ}$. Indeed, any state in $\mathcal{H}_{\rm univ}$ can be obtained (by definition) acting on the vacuum with b_{-k}, L^m_{-k} $(k \geq 2)$ oscillators and c_{-l} $(l \geq -1)$ oscillators. To compute the product $|\Psi_1\rangle * |\Psi_2\rangle$, with $\Psi_i \in \mathcal{H}_{\rm univ}$, we can use antighost and Virasoro conservation laws to move all the b, L^m oscillators out of the state-spaces (1) and (2) (where $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are inserted) onto the third state space. This is possible since we have conservation laws for b_{-k} and L^m_{-k}, when $k \geq 2$. For the ghost oscillators we can use the conservation laws for $c_0, c_{-1}, c_{-2}, \cdots$ to leave only c_1 oscillators possibly acting on spaces (1) and (2). Thus any product can be reduced to c, b, L oscillators acting on either $c_1|0\rangle * c_1|0\rangle$, $|0\rangle * c_1|0\rangle$, $c_1|0\rangle * |0\rangle$ or $|0\rangle * |0\rangle$. Since we have seen that any of those states are indeed in $\mathcal{H}_{\rm univ}$ they will remain in $\mathcal{H}_{\rm univ}$ after action by b, c and L^m oscillators. This proves $\mathcal{H}_{\rm univ}$ is a subalgebra.

Since ghost number is additive under star multiplication $(gh(A * B) = gh(A) + gh(B))$ the subspace $\mathcal{H}^{(0)}_{\rm univ}$ of ghost number zero states in $\mathcal{H}_{\rm univ}$ is itself a subalgebra. There is an even smaller universal subalgebra at ghost number zero. Consider

$$\mathcal{H}^{(0)}(L) \equiv \text{Span}\{L^{\rm tot}_{-j_1} \cdots L^{\rm tot}_{-j_p} |0\rangle\, , \, j_i \geq 2\}\, . \tag{6.37}$$

Here $L^{\rm tot}$ denotes the combined matter and ghost $(c = 0)$ Virasoro operators. Indeed since $|0\rangle * |0\rangle \in \mathcal{H}^{(0)}(L)$ (see (6.27)) it follows from the conservation laws that any product of descendents of the vacuum will be a descendent of $|0\rangle * |0\rangle \in \mathcal{H}^{(0)}(L)$ and thus a descendent of the vacuum. This confirms $\mathcal{H}^{(0)}(L)$ is a subalgebra. It would be interesting to investigate it concretely. Note that neither matter nor ghost Virasoro descendents of the vacuum form subalgebras.

7. Concluding remarks

In this paper we have developed a computational scheme for string field theory. We hope this method will be used by physicists interested in string field theory but previously mystified by the technical complexities of the requisite computations. We believe that these conservation laws are both easy to use for low level by-hand calculations and will be straightforward to implement for high level computer calculations. Such high level computations should be relevant in the near future as we are learning how to use string field theory for non-perturbative computations.

Conservation laws can also be used for string field computations in explicit backgrounds. For example, the current conservation identities described in section 4.2 can be used to compute with oscillators associated to free bosons $i\partial X$. It would be of interest to know if $\langle V_3|$, when restricted to $\mathcal{H}_{\rm univ}$, can be written as some sort of exponential of Virasoro operators.

While we have not developed the details here, our methods should be applicable to computations in superstring field theory. The relevant string field theory is non-

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polynomial [37], but since no antighost nor picture changing operators need to be inserted on the world sheet, the conservation laws discussed in this paper apply with minor modifications. It would certainly be desirable to test the brane anti-brane annihilation conjecture beyond the present accuracy of about 90% [14]–[17].

The present paper was motivated by a desire to find an analytic expression for the tachyon condensate in string field theory. Since the solution is an element of the universal subalgebra $\mathcal{H}_{\rm univ}$ we were led to believe that finding such solution would require a computational scheme that used $\mathcal{H}_{\rm univ}$ and not a background dependent representation. We have developed this computational scheme, and have used it to learn about the identity element, to compute some explicit star products and to identify a subalgebra of $\mathcal{H}_{\rm univ}$. Further developments may be needed to be able to find the exact tachyon condensate. Such condensate would represent the first nontrivial analytic solution of string field theory.

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